GMAT Club Math Book

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Number Theory

Definition

Number Theory is concerned with the properties of numbers in general, and in particular integers.
As this is a huge issue we decided to divide it into smaller topics. Below is the list of Number Theory topics.

GMAT Number Types

GMAT is dealing only with ***Real Numbers***: Integers, Fractions and Irrational Numbers.

INTEGERS

Definition

Integers are defined as: all negative natural numbers $\{...,-4,-3,-2,-1\}$, zero $\{0\}$, and positive natural numbers $\{1,2,3,4,...\}$.

Note that integers do not include decimals or fractions - just whole numbers.

Even and Odd Numbers

An even number is an *integer* that is "evenly divisible" by 2, i.e., divisible by 2 without a remainder.
An even number is an integer of the form $n=2k$, where k is an integer.

An odd number is an *integer* that is not evenly divisible by 2.
An odd number is an integer of the form $n=2k+1$, where k is an integer.

Zero is an even number.

Addition / Subtraction:
even +/- even = even;
even +/- odd = odd;
odd +/- odd = even.

Multiplication:
even * even = even;
even * odd = even;
odd * odd = odd.

Division of two integers can result into an even/odd integer or a fraction.

IRRATIONAL NUMBERS

Fractions (also known as rational numbers) can be written as *terminating* (ending) or *repeating* decimals (such as 0.5, 0.76, or 0.333333....). On the other hand, all those numbers that can be written as non-terminating, non-repeating decimals are non-rational, so they are called the "irrationals". Examples would be $\sqrt{2}$ ("the square root of two") or the number pi ($\pi=$~3.14159..., from geometry). The rational and the irrationals are two totally separate number types: there is no overlap.

Putting these two major classifications, the rational numbers and the irrational, together in one set gives you the "real" numbers.

POSITIVE AND NEGATIVE NUMBERS

A positive number is a real number that is greater than zero.
A negative number is a real number that is smaller than zero.

Zero is not positive, nor negative.

Multiplication:
positive * positive = positive
positive * negative = negative
negative * negative = positive

Division:
positive / positive = positive
positive / negative = negative
negative / negative = positive

Prime Numbers

A Prime number is a natural number with exactly two distinct natural number divisors: 1 and itself. Otherwise a number is called a *composite* number. Therefore, **1 is not a prime**, since it only has one divisor, namely 1. A number $n>1$ is prime if it cannot be written as a product of two factors a and b, both of which are greater than 1: n = ab.

• **The first twenty-six prime numbers are:**
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101

• **Note: only positive numbers can be primes.**

• There are infinitely many prime numbers.

• The **only even prime** number is 2, since any larger even number is divisible by 2. Also **2 is the smallest prime**.

• **All prime numbers except 2 and 5 end in 1, 3, 7 or 9**, since numbers ending in 0, 2, 4, 6 or 8 are multiples of 2 and numbers ending in 0 or 5 are multiples of 5. Similarly, **all prime numbers above 3 are of the form $6n-1$ or $6n+1$**, because all other numbers are divisible by 2 or 3.

• Any nonzero natural number n can be factored into primes, written as a product of primes or powers of primes. Moreover, this **factorization is unique** except for a possible reordering of the factors.

• **Prime factorization:** every positive integer greater than 1 can be written as a product of one or more prime integers in a way which is unique. For instance integer n with three unique prime factors a, b, and c can be expressed as $n=a^p*b^q*c^r$, where p, q, and r are powers of a, b, and c, respectively and are ≥ 1.
Example: $4200=2^3*3*5^2*7$.

• **Verifying the primality** (checking whether the number is a prime) of a given number n can be done by trial division, that is to say dividing n by all integer numbers smaller than $\sqrt{n}$, thereby checking whether n is a multiple of $m<\sqrt{n}$.
Example: Verifying the primality of 161: $\sqrt{161}$ is little less than 13, from integers from 2 to 13, 161 is divisible by 7, hence 161 is not prime.

• If n is a positive integer greater than 1, then there is always a prime number p with $n<p<2n$.

Factors

A divisor of an *integer* n, also called a factor of n, is an *integer* which evenly divides n without leaving a remainder. In general, it is said m is a factor of n, for non-zero integers m and n, if there exists an integer k such that $n=km$.

• 1 (and -1) are divisors of every integer.

• Every integer is a divisor of itself.

• Every integer is a divisor of 0, except, by convention, 0 itself.

• Numbers divisible by 2 are called even and numbers not divisible by 2 are called odd.

• A positive divisor of n which is different from n is called a *proper divisor*.

• An integer n > 1 whose only proper divisor is 1 is called a prime number. Equivalently, one would say that a prime number is one which has exactly two factors: 1 and itself.

• Any positive divisor of n is a product of prime divisors of n raised to some power.

• If a number equals the sum of its proper divisors, it is said to be a *perfect number*.
Example: The proper divisors of 6 are 1, 2, and 3: 1+2+3=6, hence 6 is a perfect number.

There are some elementary rules:

• If a is a factor of b and a is a factor of c, then a is a factor of $(b+c)$. In fact, a is a factor of $(mb+nc)$ for all integers m and n.

• If a is a factor of b and b is a factor of c, then a is a factor of c.

• If a is a factor of b and b is a factor of a, then $a=b$ or $a=-b$.

• If a is a factor of bc, and $gcd(a,b)=1$, then a is a factor of c.

• If p is a prime number and p is a factor of ab then p is a factor of a *or* p is a factor of b.

Finding the Number of Factors of an Integer

First make prime factorization of an integer $n=a^p*b^q*c^r$, where a, b, and c are prime factors of n and p, q, and r are their powers.

The number of factors of n will be expressed by the formula $(p+1)(q+1)(r+1)$. **NOTE:** this will include 1 and n itself.

Example: Finding the number of all factors of 450: $450=2^1*3^2*5^2$

Total number of factors of 450 including 1 and 450 itself is $(1+1)*(2+1)*(2+1)=2*3*3=18$ factors.

Finding the Sum of the Factors of an Integer

First make prime factorization of an integer $n=a^p*b^q*c^r$, where a, b, and c are prime factors of n and p, q, and r are their powers.

The sum of factors of n will be expressed by the formula: $\frac{(a^{p+1}-1)*(b^{q+1}-1)*(c^{r+1}-1)}{(a-1)*(b-1)*(c-1)}$

Example: Finding the sum of all factors of 450: $450=2^1*3^2*5^2$

The sum of all factors of 450 is $\frac{(2^{1+1}-1)*(3^{2+1}-1)*(5^{2+1}-1)}{(2-1)*(3-1)*(5-1)}=\frac{3*26*124}{1*2*4}=1209$

Greatest Common Factor (Divisor) - GCF (GCD)

The greatest common divisor (GCD), also known as the greatest common factor (GCF), or highest common factor (HCF), of two or more non-zero integers, is the largest positive integer that divides the numbers without a remainder.

To find the GCF, you will need to do prime-factorization. Then, multiply the common factors (pick the lowest power of the common factors).

• Every common divisor of a and b is a divisor of GCD (a, b).
• a*b=GCD(a, b)*lcm(a, b)

Lowest Common Multiple - LCM

The lowest common multiple or lowest common multiple (lcm) or smallest common multiple of two integers a and b is the smallest positive integer that is a multiple both of a and of b. Since it is a multiple, it can be divided by a and b without a remainder. If either a or b is 0, so that there is no such positive integer, then lcm(a, b) is defined to be zero.

To find the LCM, you will need to do prime-factorization. Then multiply all the factors (pick the highest power of the common factors).

Perfect Square

A perfect square, is an integer that can be written as the square of some other integer. For example 16=4^2, is an perfect square.

There are some tips about the perfect square:
• The number of distinct factors of a perfect square is ALWAYS ODD.
• The sum of distinct factors of a perfect square is ALWAYS ODD.
• A perfect square ALWAYS has an ODD number of Odd-factors, and EVEN number of Even-factors.
• Perfect square always has even number of powers of prime factors.

Divisibility Rules

2 - If the last digit is even, the number is divisible by 2.

3 - If the sum of the digits is divisible by 3, the number is also.

4 - If the last two digits form a number divisible by 4, the number is also.

5 - If the last digit is a 5 or a 0, the number is divisible by 5.

6 - If the number is divisible by both 3 and 2, it is also divisible by 6.

7 - Take the last digit, double it, and subtract it from the rest of the number, if the answer is divisible by 7 (including 0), then the number is divisible by 7.

8 - If the last three digits of a number are divisible by 8, then so is the whole number.

9 - If the sum of the digits is divisible by 9, so is the number.

10 - If the number ends in 0, it is divisible by 10.

11 - If you sum every second digit and then subtract all other digits and the answer is: 0, or is divisible by 11, then the number is divisible by 11.
Example: to see whether 9,488,699 is divisible by 11, sum every second digit: 4+8+9=21, then subtract the sum of

other digits: 21-(9+8+6+9)=-11, -11 is divisible by 11, hence 9,488,699 is divisible by 11.

12 - If the number is divisible by both 3 and 4, it is also divisible by 12.

25 - Numbers ending with 00, 25, 50, or 75 represent numbers divisible by 25.

Factorials

Factorial of a **positive integer** n, denoted by $n!$, is the product of all positive integers less than or equal to n. For instance $5!=1*2*3*4*5$.

- **Note: 0!=1.**
- **Note: factorial of negative numbers is undefined.**

Trailing zeros:
Trailing zeros are a sequence of 0's in the decimal representation (or more generally, in any positional representation) of a number, after which no other digits follow.

125000 has 3 trailing zeros;

The number of trailing zeros in the decimal representation of **n!**, the factorial of a non-negative integer n, can be determined with this formula:

$\frac{n}{5}+\frac{n}{5^2}+\frac{n}{5^3}+...+\frac{n}{5^k}$, where k must be chosen such that $5^k<n$.

It's easier if you look at an example:

How many zeros are in the end (after which no other digits follow) of $32!$?
$\frac{32}{5}+\frac{32}{5^2}=6+1=7$ (denominator must be less than 32, $5^2=25$ is less)

Hence, there are 7 zeros in the end of 32!

The formula actually counts the number of factors 5 in n!, but since there are at least as many factors 2, this is equivalent to the number of factors 10, each of which gives one more trailing zero.

Finding the number of powers of a prime number p, in the $n!$.

The formula is:
$\frac{n}{p}+\frac{n}{p^2}+\frac{n}{p^3}$ ... till $p^x<n$

What is the power of 2 in 25!?
$\frac{25}{2}+\frac{25}{4}+\frac{25}{8}+\frac{25}{16}=12+6+3+1=22$

Finding the power of non-prime in n!:

How many powers of 900 are in 50!

Make the prime factorization of the number: $900=2^2*3^2*5^2$, then find the powers of these prime numbers in the n!.

Find the power of 2:
$\frac{50}{2}+\frac{50}{4}+\frac{50}{8}+\frac{50}{16}+\frac{50}{32}=25+12+6+3+1=47$

= 2^{47}

Find the power of 3:

$\frac{50}{3}+\frac{50}{9}+\frac{50}{27}=16+5+1=22$

$=3^{22}$

Find the power of 5:
$\frac{50}{5}+\frac{50}{25}=10+2=12$

$=5^{12}$

We need all the prime {2,3,5} to be represented twice in 900, 5 can provide us with only 6 pairs, thus there is 900 in the power of 6 in 50!.

Consecutive Integers

Consecutive integers are integers that follow one another, without skipping any integers. 7, 8, 9, and -2, -1, 0, 1, are consecutive integers.

• Sum of n consecutive integers equals the mean multiplied by the number of terms, n. Given consecutive integers $\{-3,-2,-1,0,1,2\}$, $mean=\frac{-3+2}{2}=-\frac{1}{2}$, (mean equals to the average of the first and last terms), so the sum equals to $-\frac{1}{2}*6=-3$.

• If n is odd, the sum of consecutive integers is always divisible by n. Given $\{9,10,11\}$, we have $n=3$ consecutive integers. The sum of 9+10+11=30, therefore, is divisible by 3.

• If n is even, the sum of consecutive integers is never divisible by n. Given $\{9,10,11,12\}$, we have $n=4$ consecutive integers. The sum of 9+10+11+12=42, therefore, is not divisible by 4.

• The product of n consecutive integers is always divisible by $n!$.
Given $n=4$ consecutive integers: $\{3,4,5,6\}$. The product of 3*4*5*6 is 360, which is divisible by 4!=24.

Evenly Spaced Set

Evenly spaced set or an arithmetic progression is a sequence of numbers such that the difference of any two successive members of the sequence is a constant. The set of integers $\{9,13,17,21\}$ is an example of evenly spaced set. Set of consecutive integers is also an example of evenly spaced set.

• If the first term is a_1 and the common difference of successive members is d, then the n_{th} term of the sequence is given by:
$a_n=a_1+d(n-1)$

• In any evenly spaced set the arithmetic **mean (average) is equal to the median** and can be calculated by the formula $mean=median=\frac{a_1+a_n}{2}$, where a_1 is the first term and a_n is the last term. Given the set $\{7,11,15,19\}$, $mean=median=\frac{7+19}{2}=13$.

• The sum of the elements in any evenly spaced set is given by:
$Sum=\frac{a_1+a_n}{2}*n$, the mean multiplied by the number of terms. OR, $Sum=\frac{2a_1+d(n-1)}{2}*n$

• **Special cases:**

Sum of n first positive integers: $1+2+...+n=\frac{1+n}{2}*n$

Sum of n first positive odd numbers: $a_1+a_2+...+a_n=1+3+...+a_n=n^2$, where a_n is the last, n_{th} term and given by: $a_n=2n-1$. Given $n=5$ first odd positive integers, then their sum equals to $1+3+5+7+9=5^2=25$.

Sum of n first positive even numbers: $a_1+a_2+...+a_n=2+4+...+a_n=n(n+1)$, where a_n is the last, n_{th} term and given by: $a_n=2n$. Given $n=4$ first positive even integers, then their sum equals to $2+4+6+8=4(4+1)=20$.

• If the evenly spaced set contains odd number of elements, the mean is the middle term, so the sum is middle term multiplied by number of terms. There are five terms in the set {1, 7, 13, 19, 25}, middle term is 13, so the sum is 13*5 =65.

FRACTIONS

Definition

Fractional numbers are ratios (divisions) of integers. In other words, a fraction is formed by dividing one integer by another integer. Set of Fraction is a subset of the set of Rational Numbers.

Fraction can be expressed in two forms *fractional representation* $\left(\frac{m}{n}\right)$ and *decimal representation* $(a.bcd)$.

Fractional representation

Fractional representation is a way to express numbers that fall in between integers (note that integers can also be expressed in fractional form). A fraction expresses a part-to-whole relationship in terms of a numerator (the part) and a denominator (the whole).

• The number on top of the fraction is called *numerator* or *nominator*. The number on bottom of the fraction is called *denominator*. In the fraction, $\frac{9}{7}$, 9 is the numerator and 7 is denominator.

• Fractions that have a value between 0 and 1 are called *proper fraction*. The numerator is always smaller than the denominator. $\frac{1}{3}$ is a proper fraction.

• Fractions that are greater than 1 are called *improper fraction*. Improper fraction can also be written as a mixed number. $\frac{5}{2}$ is improper fraction.

• An integer combined with a proper fraction is called *mixed number*. $4\frac{3}{5}$ is a mixed number. This can also be written as an improper fraction: $\frac{23}{5}$

Converting Improper Fractions

• Converting Improper Fractions to Mixed Fractions:
1. Divide the numerator by the denominator
2. Write down the whole number answer
3. Then write down any remainder above the denominator

Example #1: Convert $\frac{11}{4}$ to a mixed fraction.

Solution: Divide $\frac{11}{4}=2$ with a remainder of 3. Write down the 2 and then write down the remainder 3 above the denominator 4, like this: $2\frac{3}{4}$

• Converting Mixed Fractions to Improper Fractions:
1. Multiply the whole number part by the fraction's denominator
2. Add that to the numerator
3. Then write the result on top of the denominator

Example #2: Convert $3\frac{2}{5}$ to an improper fraction.
Solution: Multiply the whole number by the denominator: $3*5=15$. Add the numerator to that: $15+2=17$. Then write that down above the denominator, like this: $\frac{17}{5}$

Reciprocal

Reciprocal for a number x, denoted by $\frac{1}{x}$ or x^{-1}, is a number which when multiplied by x yields 1. The reciprocal of a fraction $\frac{a}{b}$ is $\frac{b}{a}$. To get the reciprocal of a number, divide 1 by the number. For example reciprocal of 3 is $\frac{1}{3}$, reciprocal of $\frac{5}{6}$ is $\frac{6}{5}$.

Operation on Fractions

• **Adding/Subtracting fractions:**

To add/subtract fractions with the same denominator, add the numerators and place that sum over the common denominator.

To add/subtract fractions with the different denominator, find the Least Common Denominator (LCD) of the fractions, rename the fractions to have the LCD and add/subtract the numerators of the fractions

• **Multiplying fractions:** To multiply fractions just place the product of the numerators over the product of the denominators.

• **Dividing fractions:** Change the divisor into its reciprocal and then multiply.

Example #1: $\frac{3}{7}+\frac{2}{3}=\frac{9}{21}+\frac{14}{21}=\frac{23}{21}$

Example #2: Given $\frac{\frac{3}{5}}{2}$, take the reciprocal of 2. The reciprocal is $\frac{1}{2}$. Now multiply: $\frac{3}{5}*\frac{1}{2}=\frac{3}{10}$.

Decimal Representation

The decimals has ten as its base. Decimals can be *terminating* (ending) (such as 0.78, 0.2) or *repeating* (recurring) decimals (such as 0.333333....).

Reduced fraction $\frac{a}{b}$ (meaning that fraction is already reduced to its lowest term) can be expressed as terminating decimal *if and only* b (denominator) is of the form 2^n5^m, where m and n are non-negative integers. For example: $\frac{7}{250}$ is a terminating decimal 0.028, as 250 (denominator) equals to $2*5^3$. Fraction $\frac{3}{30}$ is also a terminating decimal, as $\frac{3}{30}=\frac{1}{10}$ and denominator $10=2*5$.

Converting Decimals to Fractions

• To convert a terminating decimal to fraction:
1. Calculate the total numbers after decimal point
2. Remove the decimal point from the number
3. Put 1 under the denominator and annex it with "0" as many as the total in step 1
4. Reduce the fraction to its lowest terms

Example: Convert 0.56 to a fraction.
1: Total number after decimal point is 2.
2 and 3: $\frac{56}{100}$.
4: Reducing it to lowest terms: $\frac{56}{100}=\frac{14}{25}$

• To convert a recurring decimal to fraction:
1. Separate the recurring number from the decimal fraction
2. Annex denominator with "9" as many times as the length of the recurring number
3. Reduce the fraction to its lowest terms

Example #1: Convert $0.393939...$ to a fraction.
1: The recurring number is 39.
2: $\frac{39}{99}$, the number 39 is of length 2 so we have added two nines.
3: Reducing it to lowest terms: $\frac{39}{99}=\frac{13}{33}$.

• To convert a mixed-recurring decimal to fraction:
1. Write down the number consisting with non-repeating digits and repeating digits.
2. Subtract non-repeating number from above.
3. Divide 1-2 by the number with 9's and 0's: for every repeating digit write down a 9, and for every non-repeating digit write down a zero after 9's.

Example #2: Convert $0.2512(12)$ to a fraction.
1. The number consisting with non-repeating digits and repeating digits is 2512;
2. Subtract 25 (non-repeating number) from above: 2512-25=2487;
3. Divide 2487 by 9900 (two 9's as there are two digits in 12 and 2 zeros as there are two digits in 25): 2487/9900=829/3300.

Rounding

Rounding is simplifying a number to a certain place value. To round the decimal drop the extra decimal places, and if the first dropped digit is 5 or greater, round up the last digit that you keep. If the first dropped digit is 4 or smaller, round down (keep the same) the last digit that you keep.

Example:
5.3485 rounded to the nearest tenth = 5.3, since the dropped 4 is less than 5.
5.3485 rounded to the nearest hundredth = 5.35, since the dropped 8 is greater than 5.
5.3485 rounded to the nearest thousandth = 5.349, since the dropped 5 is equal to 5.

Ratios and Proportions

Given that $\frac{a}{b}=\frac{c}{d}$, where a, b, c and d are non-zero real numbers, we can deduce other proportions by simple Algebra. These results are often referred to by the names mentioned along each of the properties obtained.

$\frac{b}{a}=\frac{d}{c}$ - *invertendo*

$\frac{a}{c}=\frac{b}{d}$ - *alternendo*

$\frac{a+b}{b}=\frac{c+d}{d}$ - *componendo*

$\frac{a-b}{b}=\frac{c-d}{d}$ - *dividendo*

$\frac{a+b}{a-b}=\frac{c+d}{c-d}$ - *componendo & dividendo*

EXPONENTS

Exponents are a "shortcut" method of showing a number that was multiplied by itself several times. For instance, number a multiplied n times can be written as a^n, where a represents the base, the number that is multiplied by itself n times and n represents the exponent. The exponent indicates how many times to multiple the base, a, by itself.

Exponents one and zero:
$a^0=1$ Any nonzero number to the power of 0 is 1.

For example: $5^0=1$ and $(-3)^0=1$
• **Note: the case of 0^0 is not tested on the GMAT.**

$a^1=a$ Any number to the power 1 is itself.

Powers of zero:
If the exponent is positive, the power of zero is zero: $0^n=0$, where $n>0$.

If the exponent is negative, the power of zero (0^n, where $n<0$) is undefined, because division by zero is implied.

Powers of one:
$1^n=1$ The integer powers of one are one.

Negative powers:
$a^{-n}=\frac{1}{a^n}$

Powers of minus one:
If n is an even integer, then $(-1)^n=1$.

If n is an odd integer, then $(-1)^n=-1$.

Operations involving the same exponents:
Keep the exponent, multiply or divide the bases
$a^n*b^n=(ab)^n$

$\frac{a^n}{b^n}=(\frac{a}{b})^n$

$(a^m)^n=a^{mn}$

$a^{m^n}=a^{(m^n)}$ and not $(a^m)^n$

Operations involving the same bases:
Keep the base, add or subtract the exponent (add for multiplication, subtract for division)
$a^n*a^m=a^{n+m}$

$\frac{a^n}{a^m}=a^{n-m}$

Fraction as power:
$a^{\frac{1}{n}}=\sqrt[n]{a}$

$a^{\frac{m}{n}}=\sqrt[n]{a^m}$

Exponential Equations:
When solving equations with *even exponents*, we must consider both positive and negative possibilities for the solutions.

For instance $a^2=25$, the two possible solutions are 5 and -5.

When solving equations with *odd exponents*, we'll have only one solution.

For instance for $a^3=8$, solution is $a=2$ and for $a^3=-8$, solution is $a=-2$.

Exponents and divisibility:
a^n-b^n is ALWAYS divisible by $a-b$.
a^n-b^n is divisible by $a+b$ if n is even.

a^n+b^n is divisible by $a+b$ if n is odd, and not divisible by a+b if n is even.

LAST DIGIT OF A PRODUCT

Last n digits of a product of integers are last n digits of the product of last n digits of these integers.

For instance last 2 digits of 845*9512*408*613 would be the last 2 digits of 45*12*8*13=5**40***1**04**=40*4=1**60**=60

Example: The last digit of 8594**5***8**9***5830**7**=5*9*7=4**5***7=3**5**=5?

LAST DIGIT OF A POWER

Determining the last digit of $(xyz)^n$:

1. Last digit of $(xyz)^n$ is the same as that of z^n;
2. Determine the cyclicity number c of z;
3. Find the remainder r when n divided by the cyclisity;
4. When $r>0$, then last digit of $(xyz)^n$ is the same as that of z^r and when $r=0$, then last digit of $(xyz)^n$ is the same as that of z^c, where c is the cyclisity number.

- Integer ending with 0, 1, 5 or 6, in the integer power k>0, has the same last digit as the base.
- Integers ending with 2, 3, 7 and 8 have a cyclicity of 4.
- Integers ending with 4 (e.g. $(xy4)^n$) have a cyclisity of 2. When n is odd $(xy4)^n$ will end with 4 and when n is even $(xy4)^n$ will end with 6.
- Integers ending with 9 (e.g. $(xy9)^n$) have a cyclisity of 2. When n is odd $(xy9)^n$ will end with 9 and when

n is even $(xy9)^n$ will end with 1.

Example: What is the last digit of 127^{39}?
Solution: Last digit of 127^{39} is the same as that of 7^{39}. Now we should determine the cyclisity of 7:

1. 7^1=7 (last digit is 7)
2. 7^2=9 (last digit is 9)
3. 7^3=3 (last digit is 3)
4. 7^4=1 (last digit is 1)
5. 7^5=7 (last digit is 7 again!)
...

So, the cyclisity of 7 is 4.

Now divide 39 (power) by 4 (cyclisity), remainder is 3.So, the last digit of 127^{39} is the same as that of the last digit of 7^{39}, is the same as that of the last digit of 7^3, which is 3.

ROOTS

Roots (or radicals) are the "opposite" operation of applying exponents. For instance x^2=16 and square root of 16=4.

General rules:

- $\sqrt{x}\sqrt{y}=\sqrt{xy}$ and $\frac{\sqrt{x}}{\sqrt{y}}=\sqrt{\frac{x}{y}}$.
- $(\sqrt{x})^n=\sqrt{x^n}$
- $x^{\frac{1}{n}}=\sqrt[n]{x}$
- $x^{\frac{n}{m}}=\sqrt[m]{x^n}$
- $\sqrt{a}+\sqrt{b}\neq\sqrt{a+b}$
- $\sqrt{x^2}=|x|$, when $x\leq 0$, then $\sqrt{x^2}=-x$ and when $x\geq 0$, then $\sqrt{x^2}=x$

• When the GMAT provides the square root sign for an even root, such as $\sqrt{x}$ or $\sqrt[4]{x}$, then the only accepted answer is the positive root.

That is, $\sqrt{25}=5$, NOT +5 or -5. In contrast, the equation $x^2=25$ has TWO solutions, +5 and -5. **Even roots have only a positive value on the GMAT.**

• Odd roots will have the same sign as the base of the root. For example, $\sqrt[3]{125}=5$ and $\sqrt[3]{-64}=-4$.

• For GMAT it's good to memorize following values:
$\sqrt{2}\approx 1.41$
$\sqrt{3}\approx 1.73$
$\sqrt{5}\approx 2.24$
$\sqrt{6}\approx 2.45$
$\sqrt{7}\approx 2.65$
$\sqrt{8}\approx 2.83$
$\sqrt{10}\approx 3.16$

PERCENT

Definition

A percentage is a way of expressing a number as a fraction of 100 (*per cent* meaning "per hundred"). It is often denoted using the percent sign, "%", or the abbreviation "pct". Since a percent is an amount per 100, percent can be represented as fractions with a denominator of 100. For example, 25% means 25 per 100, 25/100 and 350% means 350 per 100, 350/100.

• A percent can be represented as a decimal. The following relationship characterizes how percent and decimals interact. Percent Form / 100 = Decimal Form

For example: What is 2% represented as a decimal?
Percent Form / 100 = Decimal Form: 2%/100=0.02

Percent change

General formula for percent increase or decrease, (percent change):

$$Percent=\frac{Change}{Original}*100$$

Example: A company received $2 million in royalties on the first $10 million in sales and then $8 million in royalties on the next $100 million in sales. By what percent did the ratio of royalties to sales decrease from the first $10 million in sales to the next $100 million in sales?

Solution: Percent decrease can be calculated by the formula above:

$$Percent=\frac{Change}{Original}*100=$$
$$=\frac{\frac{2}{10}-\frac{8}{100}}{\frac{2}{10}}*100=60\%$$

, so the royalties decreased by 60%.

Simple Interest

Simple interest = principal * interest rate * time, where *"principal"* is the starting amount and *"rate"* is the interest rate at which the money grows per a given period of time (note: express the rate as a decimal in the formula). Time must be expressed in the same units used for time in the Rate.

Example: If $15,000 is invested at 10% simple annual interest, how much interest is earned after 9 months?
Solution: $15,000*0.1*9/12 = $1125

Compound Interest

$$Balance(final)=$$
$$=principal*(1+\frac{interest}{C})^{time*C}$$

, where C = the number of times compounded annually.

If C=1, meaning that interest is compounded once a year, then the formula will be: $Balance(final)=principal*(1+interest)^{time}$, where time is number of years.

Example: If $20,000 is invested at 12% annual interest, compounded quarterly, what is the balance after 2 year?

Solution: $Balance = 20{,}000*(1+\frac{0.12}{4})^{2*4} =$
$= 20{,}000*(1.03)^8 = 25{,}335.4$

Percentile

If someone's grade is in x_{th} percentile of the n grades, this means that $x\%$ of people out of n has the grades less than this person.

Example: Lena's grade was in the 80th percentile out of 120 grades in her class. In another class of 200 students there were 24 grades higher than Lena's. If nobody had Lena's grade, then Lena was what percentile of the two classes combined?

Solution:
Being in 80th percentile out of 120 grades means Lena outscored $120*0.8=96$ classmates.

In another class she would outscored $200-24=176$ students.

So, in combined classes she outscored $96+176=272$. As there are total of $120+200=320$ students, so Lena is in $\frac{272}{320}=0.85=85\%$, or in 85th percentile.

Practice from the GMAT Official Guide:

The Official Guide, 12th Edition: PS #10; PS #17; PS #19; PS #47; PS #55; PS #60; PS #64; PS #78; PS #92; PS #94; PS #109; PS #111; PS #115; PS #124; PS #128; PS #131; PS #151; PS #156; PS #166; PS #187; PS #193; PS #200; PS #202; PS #220; DS #2; DS #7; DS #21; DS #37; DS #48; DS #55; DS #61; DS #63; DS #78; DS #88; DS #92; DS #120; DS #138; DS #142; DS #143.

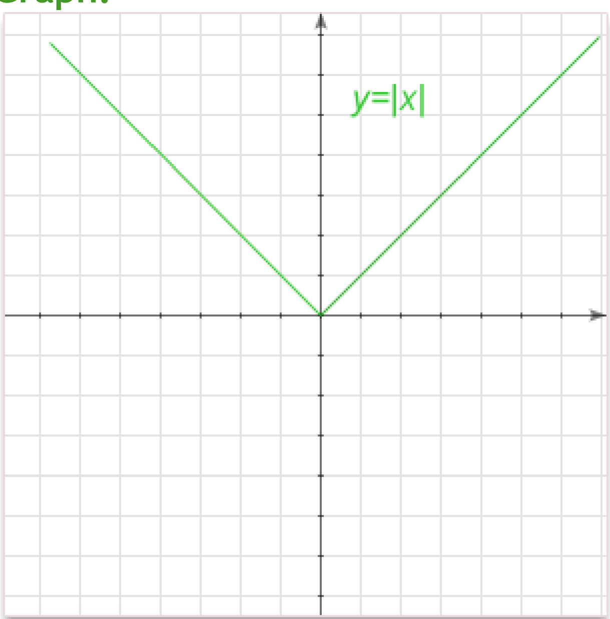
$|x|$ Absolute Value

Definition

The absolute value (or modulus) $|x|$ of a real number x is x's numerical value without regard to its sign.

For example, $|3|=3$; $|-12|=12$; $|-1.3|=1.3$

Graph:



Important properties:

$|x|\geq 0$

$|x|=\sqrt{x^2}$

$|0|=0$

$|-x|=|x|$

$|x|+|y|\geq|x+y|$

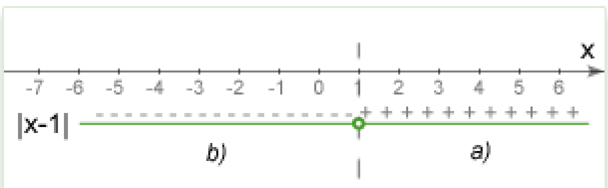
3-steps approach:

General approach to solving equalities and inequalities with absolute value:

1. Open modulus and set conditions.
To solve/open a modulus, you need to consider 2 situations to find all roots:

- Positive (or rather non-negative)
- Negative

For example, $|x-1|=4$
a) Positive: if $(x-1)\geq 0$, we can rewrite the equation as: $x-1=4$
b) Negative: if $(x-1)<0$, we can rewrite the equation as: $-(x-1)=4$
We can also think about conditions like graphics. $x=1$ is a key point in which the expression under modulus equals zero. All points right are the first conditions $(x>1)$ and all points left are second conditions $(x<1)$.



2. Solve new equations:
a) $x-1=4$ --> x=5
b) $-x+1=4$ --> x=-3

3. Check conditions for each solution:
a) $x=5$ has to satisfy initial condition $x-1>=0$. $5-1=4>0$. It satisfies. Otherwise, we would have to reject x=5.
b) $x=-3$ has to satisfy initial condition $x-1<0$. $-3-1=-4<0$. It satisfies. Otherwise, we would have to reject x=-3.

3-steps approach for complex problems

Let's consider following examples,

Example #1
Q.: $|x+3|-|4-x|=|8+x|$. How many solutions does the equation have?
Solution: There are 3 key points here: -8, -3, 4. So we have 4 conditions:

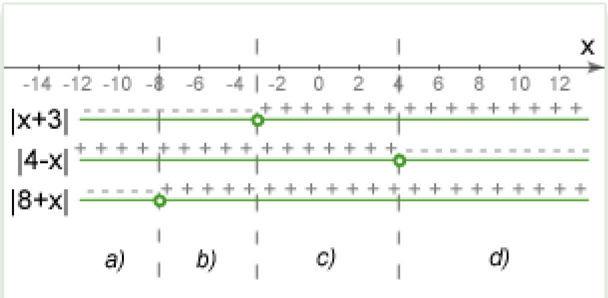
a) $x<-8$. $-(x+3)-(4-x)=-(8+x)$ --> $x=-1$. We reject the solution because our condition is not satisfied (-1 is not less than -8)

b) $-8\leq x<-3$. $-(x+3)-(4-x)=(8+x)$ --> $x=-15$. We reject the solution because our condition is not satisfied (-15 is not within (-8,-3) interval.)

c) $-3\leq x<4$. $(x+3)-(4-x)=(8+x)$ --> $x=9$. We reject the solution because our condition is not satisfied (-15 is not within (-3,4) interval.)

d) $x \geq 4$. $(x+3)+(4-x)=(8+x)$ --> $x=-1$. We reject the solution because our condition is not satisfied (-1 is not more than 4)

(Optional) The following illustration may help you understand how to open modulus at different conditions.



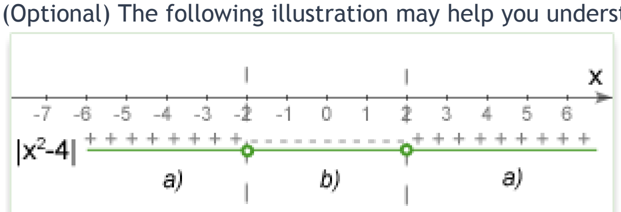
Answer: 0

Example #2
Q.: $|x^2-4|=1$. What is x?
Solution: There are 2 conditions:

a) $(x^2-4) \geq 0$ --> $x \leq -2$ or $x \geq 2$. $x^2-4=1$ --> $x^2=5$. x e $\{-\sqrt{5}, \sqrt{5}\}$ and both solutions satisfy the condition.

b) $(x^2-4)<0$ --> $-2<x<2$. $-(x^2-4)=1$ --> $x^2=3$. x e $\{-\sqrt{3}, \sqrt{3}\}$ and both solutions satisfy the condition.

(Optional) The following illustration may help you understand how to open modulus at different conditions.



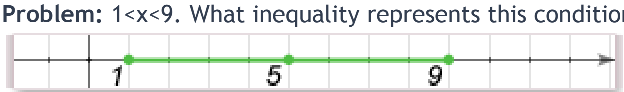
Answer: $-\sqrt{5}$, $-\sqrt{3}$, $\sqrt{3}$, $\sqrt{5}$

Tip & Tricks

The 3-steps method works in almost all cases. At the same time, often there are shortcuts and tricks that allow you to solve absolute value problems in 10-20 sec.

I. Thinking of inequality with modulus as a segment at the number line.

For example,
Problem: 1<x<9. What inequality represents this condition?



A. $|x|<3$
B. $|x+5|<4$
C. $|x-1|<9$
D. $|-5+x|<4$
E. $|3+x|<5$
Solution: 10sec. Traditional 3-steps method is too time-consume technique. First of all we find length (9-1)=8 and center (1+8/2=5) of the segment represented by $1<x<9$. Now, let’s look at our options. Only B and D has 8/2=4 on the right side and D had left site 0 at x=5. Therefore, answer is D.

II. Converting inequalities with modulus into range expression.
In many cases, especially in DS problems, it helps avoid silly mistakes.

For example,
$|x|<5$ is equal to x e (-5,5).
$|x+3|>3$ is equal to x e (-inf,-6)&(0,+inf)

III. Thinking about absolute values as distance between points at the number line.

For example,
Problem: $A<X<Y<B$. Is $|A-X| <|X-B|$?
1) $|Y-A|<|B-Y|$
Solution:

We can think about absolute values here as distance between points. Statement 1 means than distance between Y and A is less than Y and B. Because X is between A and Y, distance between $|X-A| < |Y-A|$ and at the same time distance between X and B will be larger than that between Y and B ($|B-Y|<|B-X|$). Therefore, statement 1 is sufficient.

Pitfalls

The most typical pitfall is ignoring third step in opening modulus - always check whether your solution satisfies conditions.

Practice from the GMAT Official Guide:

The Official Guide, 12th Edition: PS #22; PS #50; PS #130; DS #1; DS #153;
The Official Guide, Quantitative 2th Edition: PS #152; PS #156; DS #96; DS #120;
The Official Guide, 11th Edition: DT #9; PS #20; PS #130; DS #3; DS #105; DS #128

Algebra

Scope

Manipulation of various algebraic expressions
Equations in 1 & more variables
Dealing with non-linear equations
Algebraic identities

Notation & Assumptions

In this document, lower case roman alphabets will be used to denote variables such as *a,b,c,x,y,z,w*

In general it is assumed that the GMAT will only deal with real numbers ($\mathbb{R}$) or subsets of $\mathbb{R}$ such as Integers ($\mathbb{Z}$), rational numbers ($\mathbb{Q}$) etc.

Concept of variables

A variable is a place holder, which can be used in mathematical expressions. They are most often used for two purposes :

(a) **In Algebraic Equations** : To represent unknown quantities in known relationships. For e.g. : "Mary's age is 10 more than twice that of Jim's", we can represent the unknown "Mary's age" by x and "Jim's age" by y and then the known relationship is $x=2y+10$

(b) **In Algebraic Identities** : These are generalized relationships such as $\sqrt{x^2}=|x|$, which says for any number, if you square it and take the root, you get the absolute value back. So the variable acts like a true placeholder, which may be replaced by any number.

Basic rules of manipulation

A. When switching terms from one side to the other in an algebraic expression + becomes - and vice versa.
   E.g. $x+y=2z \Rightarrow x=2z-y$
B. When switching terms from one side to the other in an algebraic expression * becomes / and vice versa.
   E.g. $4*x=(y+1)^2 \Rightarrow x=\frac{(y+1)^2}{4}$
C. you can add/subtract/multiply/divide both sides by the same amount.
   E.g. $x+y=10z \Rightarrow \frac{x+y}{43}=\frac{10z}{43}$
D. you can take to the exponent or bring from the exponent as long as the base is the same.
   Egg 1. $x^2+2=z \Rightarrow 4^{x^2+2}=4^z$
   Egg 2. $2^{4x}=8^y \Rightarrow 2^{4x}=2^{3y} \Rightarrow 4x=3y$

It is important to note that all the operations above are possible not just with constants but also with variables themselves. So you can "add x" or "multiply with y" on both sides while maintaining the expression. But what you need to be very careful about is when dividing both sides by a variable. **When you divide both sides by a variable (or do operations like "canceling x on both sides") you implicitly assume that the variable cannot be equal to 0, as division by 0 is undefined.** This is a concept shows up very often on GMAT questions.

Degree of an expression

The degree of an algebraic expression is defined as the highest power of the variables present in the expression.
Degree 1 : Linear
Degree 2 : Quadratic
Degree 3 : Cubic
Degree 4 : Bi-quadratic

Example: $x+y$ the degree is 1
x^3+x+2 the degree is 3
x^3+z^5 the degree of x is 3, degree of z is 5, degree of the expression is 5

Solving equations of degree 1 : LINEAR

Degree 1 equations or linear equations are equations in one or more variable such that degree of each variable is one. Let us consider some special cases of linear equations :

<u>One variable</u>

Such equations will always have a solution. General form is $ax=b$ and solution is $x=(b/a)$

<u>One equation in Two variables</u>
This is not enough to determine x and y uniquely. There can be infinitely many solutions.

<u>Two equations in Two variables</u>
If you have a linear equation in 2 variables, you need at least 2 equations to solve for both variables. The general form is :
$$ax+by=c$$
$$dx+ey=f$$

If $(a/d)=(b/e)=(c/f)$ then there are infinite solutions. Any point satisfying one equation will always satisfy the second

If $(a/d)=(b/e)\neq(c/f)$ then there is no such x and y which will satisfy both equations. No solution

In all other cases, solving the equations is straight forward, multiply equation (2) by a/d and subtract from (1).

<u>More than two equations in Two variables</u>
Pick any 2 equations and try to solve them :
Case 1 : No solution --> Then there is no solution for bigger set
Case 2 : Unique solution --> Substitute in other equations to see if the solution works for all others
Case 3 : Infinite solutions --> Out of the 2 equations you picked, replace any one with an un-picked equation and repeat.

<u>More than 2 variables</u>
This is not a case that will be encountered often on the GMAT. But in general for n variables you will need at least n equations to get a unique solution. Sometimes you can assign unique values to a subset of variables using less than n equations using a small trick. For example consider the equations :
$$x+2y+5z=20$$
$$x+4y+10z=40$$
In this case you can treat $2y+z$ as a single variable to get :
$$x+(2y+5z)=20$$
$$x+2*(2y+5z)=40$$
These can be solved to get x=0 and 2y+5z=20

There is a common misconception that you need n equations to solve n variables. This is not true.

Solving equations of degree 2 : QUADRATIC

The general form of a quadratic equation is $ax^2+bx+c=0$

The equation has no solution if $b^2<4ac$

The equation has exactly one solution if $b^2=4ac$

This equation has 2 solutions given by $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ if $b^2>4ac$

The sum of roots is $\frac{-b}{a}$

The product of roots is $\frac{c}{a}$

If the roots are r_1 and r_2, the equation can be written as $(x-r_1)(x-r_2)=0$

A quick way to solve a quadratic, without the above formula is to factorize it :
Step 1> Divide throughout by coeff of x^2 to put it in the form $x^2+dx+e=0$
Step 2> Sum of roots = -d and Product = e. Search for 2 numbers which satisfy this criteria, let them be f,g
Step 3> The equation may be re-written as (x-f)(x-g)=0. And the solutions are f,g

E.g. $x^2+11x+30=0$
The sum is -11 and the product is 30. So numbers are -5,-6
$x^2+11x+30=x^2+5x+6x+30=x(x+5)+6(x+5)=(x+5)(x+6)$

Solving equations with DEGREE>2

You will never be asked to solved higher degree equations, except in some cases where using simple tricks these equations can either be factorized or be reduced to a lower degree or both. What you need to note is that an equation of degree n has at most n unique solutions.

Factorization
This is the easiest approach to solving higher degree equations. Though there is no general rule to do this, generally a knowledge of algebraic identities helps. The basic idea is that if you can write an equation in the form :

$A^*B^*C=0$

where each of A,B,C are algebraic expressions. Once this is done, the solution is obtained by equating each of A,B,C to 0 one by one.

E.g. $x^3+11x^2+30x=0$
$x^*(x^2+11x+30)=0$
$x^*(x+5)^*(x+6)=0$

So the solution is x=0,-5,-6

Reducing to lower degree

This is useful sometimes when it is easy to see that a simple variable substitution can reduce the degree.

E.g. $x^6-3x^3+2=0$

Here let $y=x^3$
$y^2-3y+2=0$
$(y-2)(y-1)=0$

So the solution is y=1,2 or x^3=1,2 or x=1,cube_root(3)

Other tricks

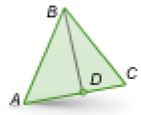
Sometimes we are given conditions such as the variables being integers which make the solutions much easier to find. When we know that the solutions are integral, often times solutions are easy to find using just brute force.

E.g. $a^2+b^2=116$ and we know a,b are integers such that a<b
We can solve this by testing values of a and checking if we can find b
a=1 b=root(115) Not integer
a=2 b=root(112) Not integer
a=3 b=root(107) Not integer
a=4 b=root(100)=10
a=5 b=root(91) Not integer
a=6 b=root(80) Not integer
a=7 b=root(67) Not integer
a=8 b=root(52)<a
So the answer is (4,10)

Algebraic Identities

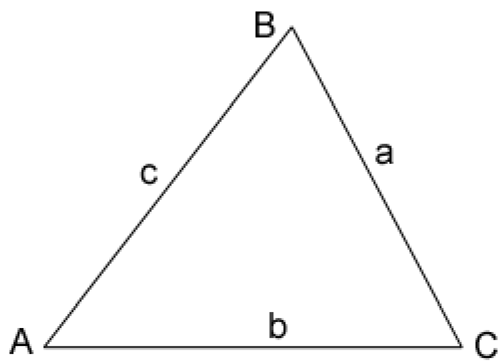
These can be very useful in simplifying & solving a lot of questions :

- $(x+y)^2=x^2+y^2+2xy$
- $(x-y)^2=x^2+y^2-2xy$
- $x^2-y^2=(x+y)(x-y)$
- $(x+y)^2-(x-y)^2=4xy$
- $(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)$
- $x^3+y^3=(x+y)(x^2+y^2-xy)$
- $x^3-y^3=(x-y)(x^2+y^2+xy)$



Triangles

Triangle *A closed figure consisting of three line segments linked end-to-end. A 3-sided polygon.*

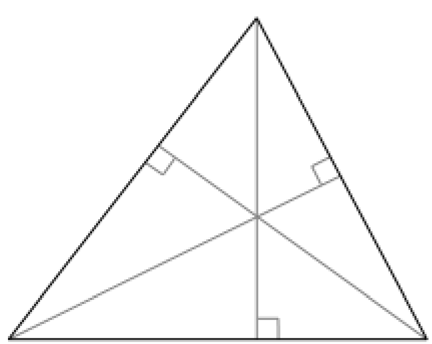


Vertex *The vertex (plural: vertices) is a corner of the triangle. Every triangle has three vertices.*

Base *The base of a triangle can be any one of the three sides, usually the one drawn at the bottom.*

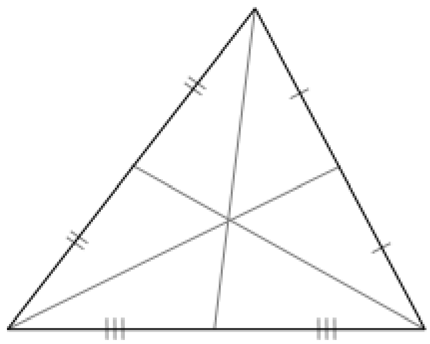
- You can pick any side you like to be the base.
- Commonly used as a reference side for calculating the area of the triangle.
- In an isosceles triangle, the base is usually taken to be the unequal side.

Altitude *The altitude of a triangle is the perpendicular from the base to the opposite vertex. (The base may need to be extended).*



- Since there are three possible bases, there are also three possible altitudes.
- The three altitudes intersect at a single point, called the orthocenter of the triangle.

Median *The median of a triangle is a line from a vertex to the midpoint of the opposite side.*



- **The three medians intersect at a single point, called the centroid of the triangle.**
- **Each median divides the triangle into two smaller triangles which have the same area.**
- Because there are three vertices, there are of course three possible medians.
- No matter what shape the triangle, all three always intersect at a single point. This point is called the **centroid** of the triangle.
- The three medians divide the triangle into six smaller triangles of equal area.
- The centroid (point where they meet) is the center of gravity of the triangle
- **Two-thirds of the length of each median is between the vertex and the centroid, while one-third is between the centroid and the midpoint of the opposite side.**
- $m=\sqrt{\frac{2b^2+2c^2-a^2}{4}}$, where a, b and c are the sides of the triangle and a is the side of the triangle whose midpoint is the extreme point of median m.

Area *The number of square units it takes to exactly fill the interior of a triangle.*

Usually called "half of base times height", the area of a triangle is given by the formula below.

- $A=\frac{hb}{2}$

Other formula:

- $A=\frac{P*r}{2}$
- $A=\frac{abc}{4R}$

Where b is the length of the base, a and c the other sides; h is the length of the corresponding altitude; R is the Radius of circumscribed circle; r is the radius of inscribed circle; P is the perimeter

- Heron's or Hero's Formula for calculating the area $A=\sqrt{s(s-a)(s-b)(s-c)}$ where a,b,c are the three sides of the triangle and $s=\frac{a+b+c}{2}$ which is the semi perimeter of the triangle.

Perimeter *The distance around the triangle. The sum of its sides.*

- **For a given perimeter equilateral triangle has the largest area.**
- **For a given area equilateral triangle has the smallest perimeter.**

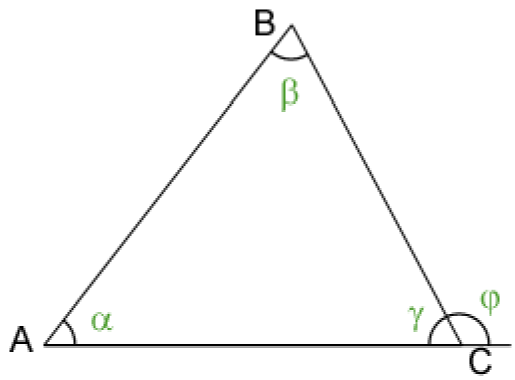
Relationship of the Sides of a Triangle

- **The length of any side of a triangle must be larger than the positive difference of the other two sides, but smaller than the sum of the other two sides.**

Interior angles *The three angles on the inside of the triangle at each vertex.*

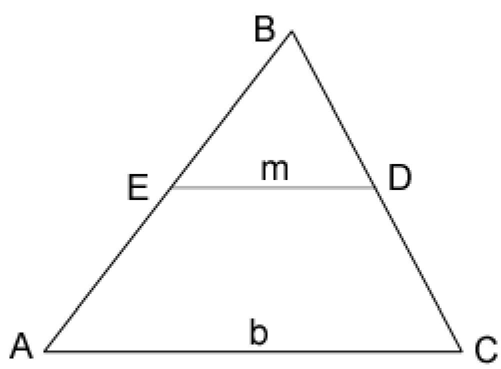
- **The interior angles of a triangle always add up to 180°**
- Because the interior angles always add to 180°, every angle must be less than 180°
- **The bisectors of the three interior angles meet at a point, called the incenter, which is the center of the incircle of the triangle.**

Exterior angles *The angle between a side of a triangle and the extension of an adjacent side.*



- **An exterior angle of a triangle is equal to the sum of the opposite interior angles.**
- If the equivalent angle is taken at each vertex, the exterior angles always add to 360° In fact, this is true for any convex polygon, not just triangles.

Midsegment of a Triangle *A line segment joining the midpoints of two sides of a triangle*



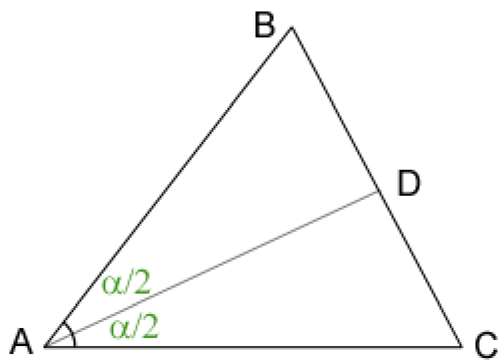
- A triangle has 3 possible midsegments.

• **The midsegment is always parallel to the third side of the triangle.**
• **The midsegment is always half the length of the third side.**
• A triangle has three possible midsegments, depending on which pair of sides is initially joined.

Relationship of sides to interior angles in a triangle

• The shortest side is always opposite the smallest interior angle
• The longest side is always opposite the largest interior angle

Angle bisector *An angle bisector divides the angle into two angles with equal measures.*



• An angle only has one bisector.
• **Each point of an angle bisector is equidistant from the sides of the angle.**
• **The angle bisector theorem states that the ratio of the length of the line segment BD to the length of segment DC is equal to the ratio of the length of side AB to the length of side AC:** $\frac{BD}{DC}=\frac{AB}{AC}$
• **The incenter is the point where the angle bisectors intersect. The incenter is also the center of the triangle's incircle - the largest circle that will fit inside the triangle.**

Similar Triangles *Triangles in which the three angles are identical.*

• It is only necessary to determine that two sets of angles are identical in order to conclude that two triangles are similar; the third set will be identical because all of the angles of a triangle always sum to 180º.
• In similar triangles, the sides of the triangles are in some proportion to one another. For example, a triangle with lengths 3, 4, and 5 has the same angle measures as a triangle with lengths 6, 8, and 10. The two triangles are similar, and all of the sides of the larger triangle are twice the size of the corresponding legs on the smaller triangle.

• **If two similar triangles have sides in the ratio** $\frac{x}{y}$**, then their areas are in the ratio** $\frac{x^2}{y^2}$

Congruence of triangles *Two triangles are congruent if their corresponding sides are equal in length and their corresponding angles are equal in size.*

1. **SAS (Side-Angle-Side):** If two pairs of sides of two triangles are equal in length, and **the included angles** are equal in measurement, then the triangles are congruent.

2. **SSS (Side-Side-Side):** If three pairs of sides of two triangles are equal in length, then the triangles are congruent.

3. **ASA (Angle-Side-Angle):** If two pairs of angles of two triangles are equal in measurement, and **the included sides** are equal in length, then the triangles are congruent.

So, knowing SAS or ASA <u>is sufficient to</u> determine unknown angles or sides.

NOTE IMPORTANT EXCEPTION:
The SSA condition (Side-Side-Angle) which specifies two sides and a non-included angle (also known as ASS, or Angle-Side-Side) does not always prove congruence, even when the equal angles are opposite equal sides.

Specifically, SSA does not prove congruence when the angle is acute and the opposite side is shorter than the known adjacent side but longer than the sine of the angle times the adjacent side. This is the ambiguous case. In all other cases with corresponding equalities, SSA proves congruence.

The SSA condition proves congruence if the angle is obtuse or right. In the case of the right angle (also known as the HL (Hypotenuse-Leg) condition or the RHS (Right-angle-Hypotenuse-Side) condition), we can calculate the third side and fall back on SSS.

To establish congruence, it is also necessary to check that the equal angles are opposite equal sides.

So, knowing two sides and non-included angle <u>is NOT sufficient</u> to calculate unknown side and angles.

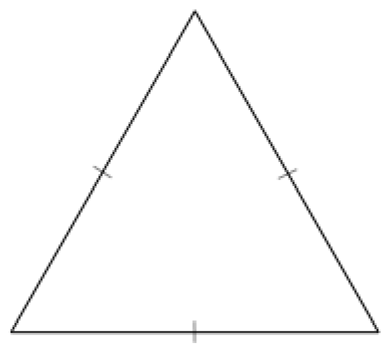
Angle-Angle-Angle
AAA (Angle-Angle-Angle) says nothing about the size of the two triangles and hence proves only similarity and not congruence.

So, knowing three angles <u>is NOT sufficient</u> to determine lengths of the sides.

Scalene triangle *all sides and angles are different from one another*

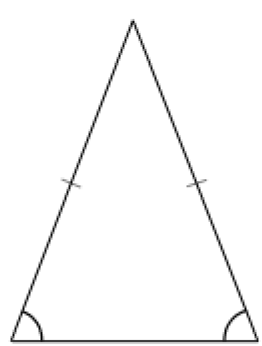
• All properties mentioned above can be applied to the scalene triangle, if not mentioned the special cases (equilateral, etc.)

Equilateral triangle *all sides have the same length.*



- **An equilateral triangle is also a regular polygon with all angles measuring 60°.**
- **The area is** $A = a^2 * \frac{\sqrt{3}}{4}$
- **The perimeter is** $P = 3a$
- **The radius of the circumscribed circle is** $R = a * \frac{\sqrt{3}}{3}$
- **The radius of the inscribed circle is** $r = a * \frac{\sqrt{3}}{6}$
- And the altitude is $h = a * \frac{\sqrt{3}}{2}$ (Where a is the length of a side.)
- For any point P within an equilateral triangle, the sum of the perpendiculars to the three sides is equal to the altitude of the triangle.
- **For a given perimeter equilateral triangle has the largest area.**
- **For a given area equilateral triangle has the smallest perimeter.**
- **With an equilateral triangle, the radius of the incircle is exactly half the radius of the circumcircle.**

Isosceles triangle *two sides are equal in length.*

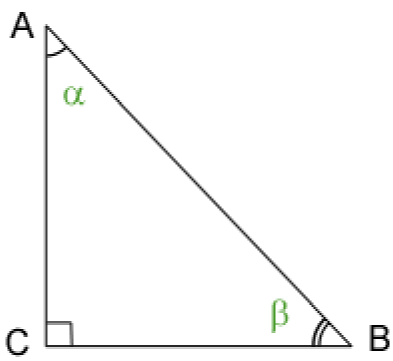


- An isosceles triangle also has two angles of the same measure; namely, the angles opposite to the two sides of the same length.
- For an isosceles triangle with given length of equal sides right triangle (included angle) has the largest area.
- **To find the base given the leg and altitude, use the formula:** $B = 2\sqrt{L^2 - A^2}$

• To find the leg length given the base and altitude, use the formula: $L=\sqrt{A^2+(\frac{B}{2})^2}$

• To find the altitude given the base and leg, use the formula: $A=\sqrt{L^2-(\frac{B}{2})^2}$ (Where: L is the length of a leg; A is the altitude; B is the length of the base)

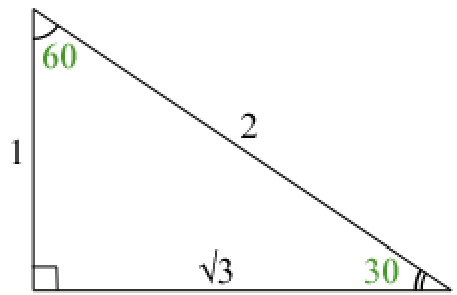
Right triangle *A triangle where one of its interior angles is a right angle (90 degrees)*



• Hypotenuse: the side opposite the right angle. This will always be the longest side of a right triangle.
• The two sides that are not the hypotenuse. They are the two sides making up the right angle itself.
• **Theorem by Pythagoras** defines the relationship between the three sides of a right triangle: $a^2+b^2=c^2$, where c is the length of the hypotenuse and a, b are the lengths of the other two sides.
• **In a right triangle, the midpoint of the hypotenuse is equidistant from the three polygon vertices**
• **A right triangle can also be isosceles if the two sides that include the right angle are equal in length (AB and AC in the figure above)**
• **Right triangle with a given hypotenuse has the largest area when it's an isosceles triangle.**
• **A right triangle can never be equilateral, since the hypotenuse (the side opposite the right angle) is always longer than the other two sides.**
• **Any triangle whose sides are in the ratio 3:4:5 is a right triangle. Such triangles that have their sides in the ratio of whole numbers are called Pythagorean Triples. There are an infinite number of them, and this is just the smallest. If you multiply the sides by any number, the result will still be a right triangle whose sides are in the ratio 3:4:5. For example 6, 8, and 10.**

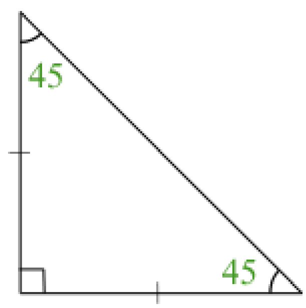
• A Pythagorean triple consists of three positive integers a, b, and c, such that $a^2+b^2=c^2$. Such a triple is commonly written (a,b,c), and a well-known example is $(3,4,5)$. **If** (a,b,c) **is a Pythagorean triple, then so is** (ka,kb,kc) **for any positive integer** k **.** There are 16 primitive Pythagorean triples with c ≤ 100:
(3, 4, 5) (5, 12, 13) (7, 24, 25) (8, 15, 17) (9, 40, 41) (11, 60, 61) (12, 35, 37) (13, 84, 85) (16, 63, 65) (20, 21, 29) (28, 45, 53) (33, 56, 65) (36, 77, 85) (39, 80, 89) (48, 55, 73) (65, 72, 97).

• **A right triangle where the angles are 30°, 60°, and 90°.**



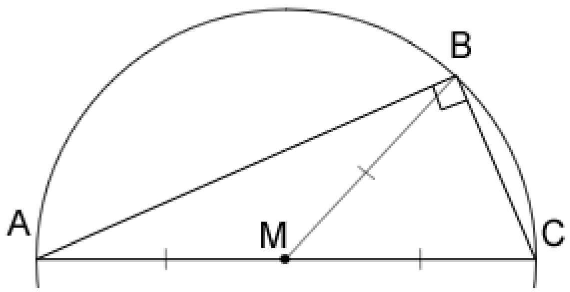
This is one of the 'standard' triangles you should be able recognize on sight. A fact you should commit to memory is: The sides are always in the ratio $1:\sqrt{3}:2$.
Notice that the smallest side (1) is opposite the smallest angle (30°), and the longest side (2) is opposite the largest angle (90°).

• **A right triangle where the angles are 45°, 45°, and 90°.**



This is one of the 'standard' triangles you should be able recognize on sight. A fact you should also commit to memory is: The sides are always in the ratio $1:1:\sqrt{2}$. With the $\sqrt{2}$ being the hypotenuse (longest side). This can be derived from Pythagoras' Theorem. Because the base angles are the same (both 45°) the two legs are equal and so the triangle is also isosceles.
• Area of a 45-45-90 triangle. As you see from the figure above, two 45-45-90 triangles together make a square, so the area of one of them is half the area of the square. As a formula $A = \frac{S^2}{2}$. Where S is the length of either short side.

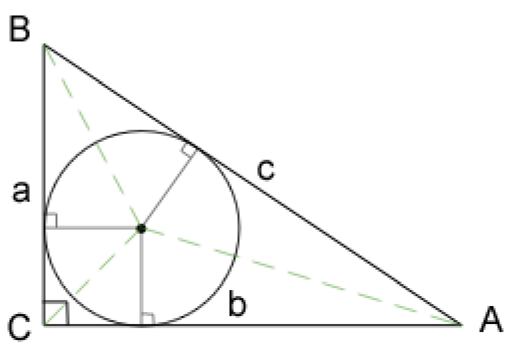
• **Right triangle inscribed in circle:**



$R = \frac{AC}{2}$

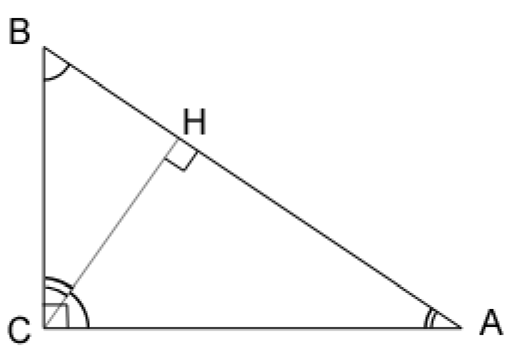
• If M is the midpoint of the hypotenuse, then $BM = \frac{1}{2}AC$. One can also say that point B is located on the circle with diameter AC. Conversely, if B is any point of the circle with diameter AC (except A or C themselves) then angle B in triangle ABC is a right angle.
• A right triangle inscribed in a circle must have its hypotenuse as the diameter of the circle. The reverse is also true: if the diameter of the circle is also the triangle's hypotenuse, then that triangle is a right triangle.

• **Circle inscribed in right triangle:** $r = \frac{ab}{a+b+c} = \frac{a+b-c}{2}$



Note that in picture above the right angle is C.

• Given a right triangle, draw the altitude from the right angle.



Then the triangles ABC, CHB and CHA are similar. Perpendicular to the hypotenuse will always divide the triangle into two triangles with the same properties as the original triangle.

Practice from the GMAT Official Guide:

The Official Guide, 12th Edition: DT #19; DT #28; PS #48; PS #152; PS #205; PS #209; PS #229; DS #20; DS #56; DS #74; DS #109; DS #140; DS #144; DS #149; DS #157; DS #160; DS #173;
The Official Guide, Quantitative 2th Edition: PS #44; PS #71; PS #85; PS #145; PS #157; PS #162; DS #19; DS #65; DS #88; DS #91; DS #123;
The Official Guide, 11th Edition: DT #19; DT #28; PS #45; PS #152; PS #158; PS #226; PS #248; DS #27; DS #32; DS #51; DS #66; DS #108; DS #113; DS #124; DS #136; DS #152.

Polygons

Types of Polygon

Regular A polygon with all sides and interior angles the same. Regular polygons are always convex.

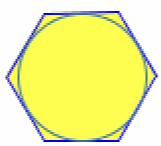
Convex All interior angles less than 180°, and all vertices 'point outwards' away from the interior. The opposite of concave. Regular polygons are always convex.

Definitions, Properties and Tips

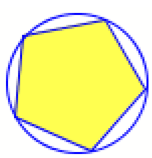
• **Sum of Interior Angles** $180(n-2)$ where n is the number of sides

• For a regular polygon, the total described above is spread evenly among all the interior angles, since they all have the same values. So for example the interior angles of a pentagon always add up to 540°, so in a regular pentagon (5 sides), each one is one fifth of that, or 108°. Or, as a formula, each interior angle of a regular polygon is given by: $\frac{180(n-2)}{n}$, where n is the number of sides.

• The apothem of a polygon is a line from the center to the midpoint of a side. This is also the inradius - the radius of the incircle.

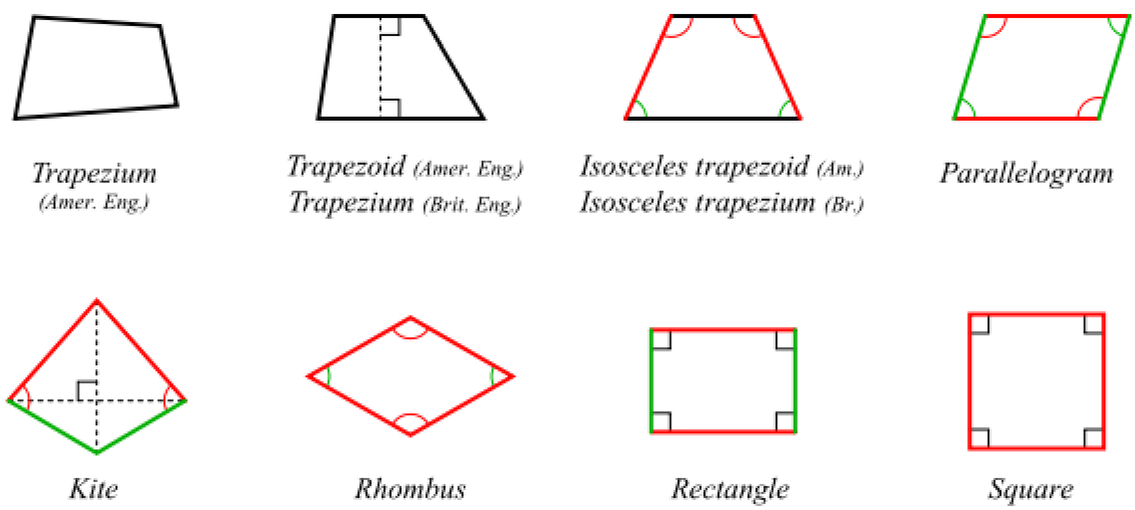


• The radius of a regular polygon is a line from the center to any vertex. It is also the radius of the circumcircle of the polygon.



GMAT is dealing mainly with the following polygons:

Quadrilateral *A polygon with four 'sides' or edges and four vertices or corners.*



Types of quadrilateral:

Square All sides equal, all angles 90°. See Definition of a square.

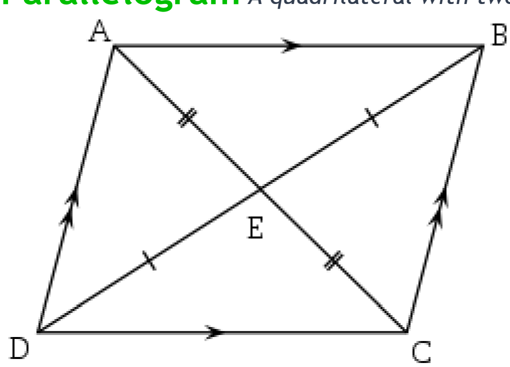
Rectangle Opposite sides equal, all angles 90°. See Definition of a rectangle.

Parallelogram Opposite sides parallel. See Definition of a parallelogram.

Trapezoid Two sides parallel. See Definition of a trapezoid.

Rhombus Opposite sides parallel and equal. See Definition of a rhombus.

Parallelogram *A quadrilateral with two pairs of parallel sides.*



Properties and Tips

- **Opposite sides of a parallelogram are equal in length.**
- **Opposite angles of a parallelogram are equal in measure.**
- Opposite sides of a parallelogram will never intersect.

- **The diagonals of a parallelogram bisect each other.**
- **Consecutive angles are supplementary, add to 180°.**
- **The area**, A, of a parallelogram is $A = bh$, where b is the base of the parallelogram and h is its height.
- The area of a parallelogram is twice the area of a triangle created by one of its diagonals.

A parallelogram is a quadrilateral with opposite sides parallel and congruent. It is the "parent" of some other quadrilaterals, which are obtained by adding restrictions of various kinds:
- A rectangle is a parallelogram but with all angles fixed at 90°
- A rhombus is a parallelogram but with all sides equal in length
- A square is a parallelogram but with all sides equal in length and all angles fixed at 90°

Rectangle *A 4-sided polygon where all interior angles are 90°*

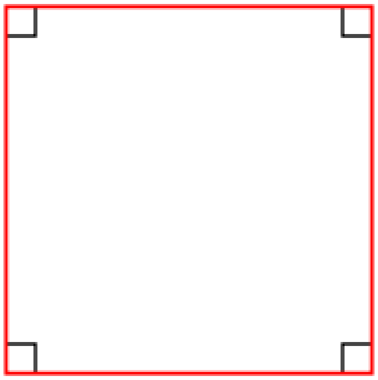
Properties and Tips

- **Opposite sides are parallel and congruent**
- **The diagonals bisect each other**
- **The diagonals are congruent**
- **A square is a special case of a rectangle where all four sides are the same length.**
- **It is also a special case of a parallelogram but with extra limitation that the angles are fixed at 90°.**
- **The two diagonals are congruent (same length).**
- **Each diagonal bisects the other. In other words, the point where the diagonals intersect (cross), divides each diagonal into two equal parts.**
- **Each diagonal divides the rectangle into two congruent right triangles. Because the triangles are congruent, they have the same area, and each triangle has half the area of the rectangle.**
- $Diagonal = \sqrt{w^2+h^2}$ where: w is the width of the rectangle, **h** is the height of the rectangle.
- The **area** of a rectangle is given by the formula $Width * Height$.

A rectangle can be thought about in other ways:

- A square is a special case of a rectangle where all four sides are the same length. Adjust the rectangle above to create a square.
- It is also a special case of a parallelogram but with extra limitation that the angles are fixed at 90°.

Squares *A 4-sided regular polygon with all sides equal and all internal angles 90°*



Properties and Tips

• If the diagonals of a rhombus are equal, then that rhombus must be a square. The diagonals of a square are (about 1.414) times the length of a side of the square.
• A square can also be defined as a rectangle with all sides equal, or a rhombus with all angles equal, or a parallelogram with equal diagonals that bisect the angles.
• If a figure is both a rectangle (right angles) and a rhombus (equal edge lengths), then it is a square. (Rectangle (four equal angles) + Rhombus (four equal sides) = Square)

• If a circle is circumscribed around a square, the area of the circle is $\frac{\pi}{2}$ (about 1.57) times the area of the square.

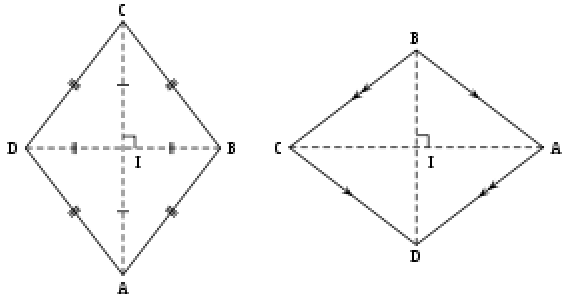
• If a circle is inscribed in the square, the area of the circle is $\frac{\pi}{4}$ (about 0.79) times the area of the square.
• A square has a larger area than any other quadrilateral with the same perimeter.
• Like most quadrilaterals, the **area** is the length of one side times the perpendicular height. So in a square this is simply: $area = s^2$, where s is the length of one side.
• **The "diagonals" method.** If you know the lengths of the diagonals, the area is half the product of the diagonals.

Since both diagonals are congruent (same length), this simplifies to: $area = \frac{d^2}{2}$, where d is the length of either diagonal
• Each diagonal of a square is the perpendicular bisector of the other. That is, each cuts the other into two equal parts, and they cross and right angles (90°).
• **The length of each diagonal** is $s\sqrt{2}$ where s is the length of any one side.

A square is both a rhombus (equal sides) and a rectangle (equal angles) and therefore has all the properties of both these shapes, namely:
The diagonals of a square bisect each other.
• The diagonals of a square bisect its angles.
• The diagonals of a square are perpendicular.
• Opposite sides of a square are both parallel and equal.
• All four angles of a square are equal. (Each is 360/4 = 90 degrees, so every angle of a square is a right angle.)
• The diagonals of a square are equal.

A square can be thought of as a special case of other quadrilaterals, for example
• a rectangle but with adjacent sides equal
• a parallelogram but with adjacent sides equal and the angles all 90°
• a rhombus but with angles all 90°

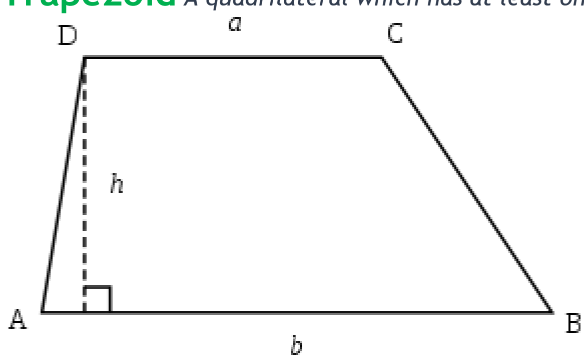
Rhombus *A quadrilateral with all four sides equal in length.*



Properties and Tips

• **A rhombus is actually just a special type of parallelogram.** Recall that in a parallelogram each pair of opposite sides are equal in length. With a rhombus, all four sides are the same length. **It therefore has all the properties of a parallelogram.**
• **The diagonals of a rhombus always bisect each other at 90˚.**
• There are several ways to find the **area of a rhombus**. The most common is: $base*altitude$.
• **The "diagonals" method.** Another simple formula for the area of a rhombus when you know the lengths of the diagonals. The area is half the product of the diagonals. As a formula: $\frac{d1*d2}{2}$, where $d1$ is the length of a diagonal $d2$ is the length of the other diagonal.

Trapezoid *A quadrilateral which has at least one pair of parallel sides.*



Properties and Tips

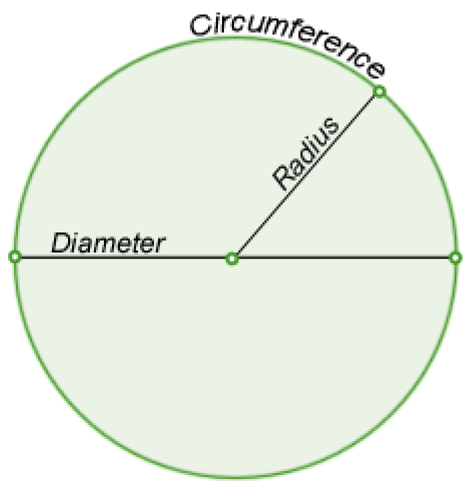
• **Base** - One of the parallel sides. Every trapezoid has two bases.
• **Leg** - The non-parallel sides are legs. Every trapezoid has two legs.
• **Altitude** - The altitude of a trapezoid is the perpendicular distance from one base to the other. (One base may need to be extended).
• If both legs are the same length, this is called an isosceles trapezoid, and both base angles are the same.
• **If the legs are parallel, it now has two pairs of parallel sides, and is a parallelogram.**
• **Median** - The median of a trapezoid is a line joining the midpoints of the two legs.
• **The median line is always parallel to the bases.**
• **The length of the median is the average length of the bases, or using the formula:** $\frac{AB+DC}{2}$
• **The median line is halfway between the bases.**
• **The median divides the trapezoid into two smaller trapezoids each with half the altitude of the original.**

• **Area** - The usual way to calculate the area is the average base length times altitude. The area of a trapezoid is given by the formula $h*\frac{a+b}{2}$ where
a, b are the lengths of the two bases h is the altitude of the trapezoid
• The **area** of a trapezoid is the $altitude*median$.

Circles

Definition

A line forming a closed loop, every point on which is a fixed distance from a center point. Circle could also be defined as the set of all points equidistant from the center.



Center -a point inside the circle. All points on the circle are equidistant (same distance) from the center point.

Radius - the distance from the center to any point on the circle. It is half the diameter.

Diameter -t he distance across the circle. The length of any chord passing through the center. It is twice the radius.

Circumference - the distance around the circle.

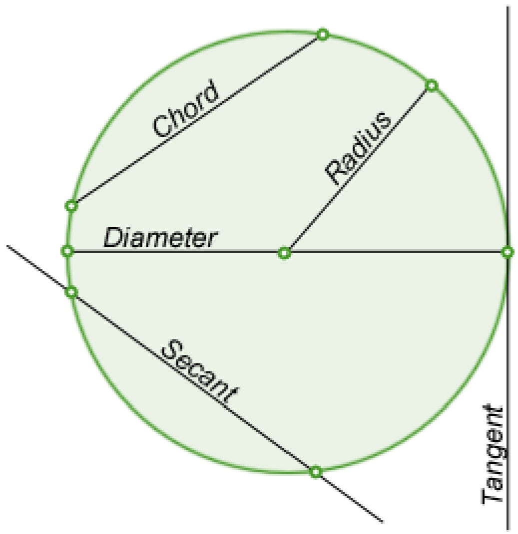
Area - strictly speaking a circle is a line, and so has no area. What is usually meant is the area of the region enclosed by the circle.

Chord - line segment linking any two points on a circle.

Tangent -a line passing a circle and touching it at just one point.
The tangent line is always at the 90 degree angle (perpendicular) to the radius of a circle.

Secant A line that intersects a circle at two points.

π In any circle, if you divide the circumference (distance around the circle) by its diameter (distance across the circle), you always get the same number. This number is called Pi and is approximately 3.142.



• A circle is the shape with the largest area for a given length of perimeter (has the highest area to length ratio when compared to other geometric figures such as triangles or rectangles)
• All circles are similar
• To form a unique circle, it needs to have 3 points which are not on the same line.

Circumference, Perimeter of a circle

Given a radius r of a circle, the circumference can be calculated using the formula: $Circumference = 2\pi r$

If you know the diameter D of a circle, the circumference can be found using the formula: $Circumference = \pi D$

If you know the area A of a circle, the circumference can be found using the formula: $Circumference = \sqrt{4\pi A}$

Area enclosed by a circle

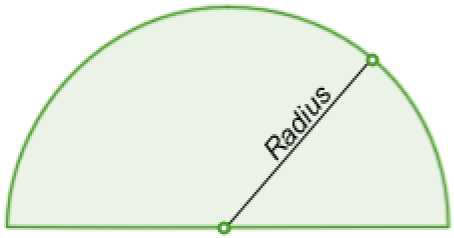
Given the radius r of a circle, the area can be calculated using the formula: $Area = \pi r^2$

If you know the diameter D of a circle, the area can be found using the formula: $Area = \frac{\pi D^2}{4}$

If you know the circumference C of a circle, the area can be found using the formula: $Area = \frac{C^2}{4\pi}$

Semicircle

Half a circle or a closed shape consisting of half a circle and a diameter of that circle.

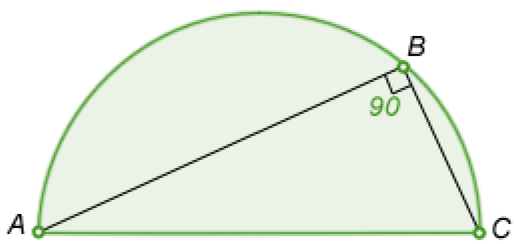


• The area of a semicircle is half the area of the circle from which it is made: $Area = \frac{\pi r^2}{2}$

• The perimeter of a semicircle **is not** half the perimeter of a circle. From the figure above, you can see that the perimeter is the curved part, which is half the circle, plus the diameter line across the bottom. So, the formula for the perimeter of a semicircle is: $Perimeter = \pi r + 2r = r(\pi + 2)$

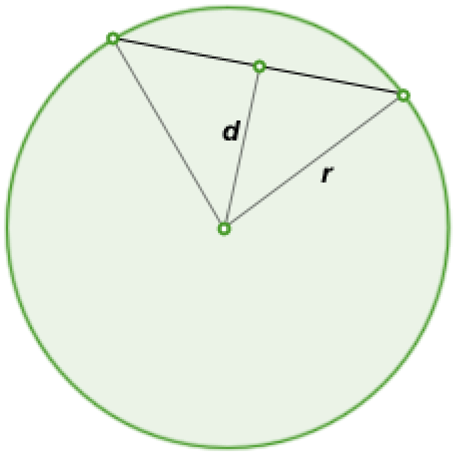
• The angle inscribed in a semicircle is always 90°.

• Any diameter of a circle subtends a right angle to any point on the circle. No matter where the point is, the triangle formed with diameter is always a right triangle.



Chord

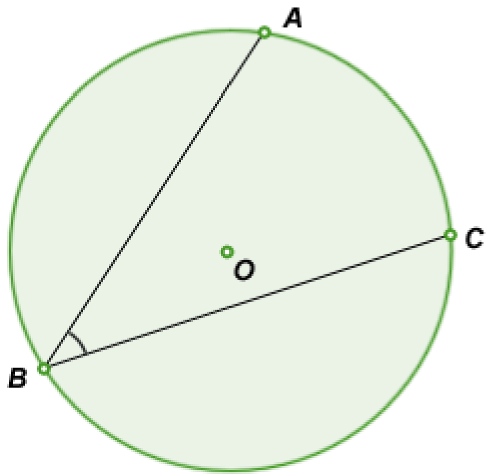
A line that links two points on a circle or curve.



• A diameter is a chord that contains the center of the circle.
• Below is a formula for the length of a chord if you know the radius and the perpendicular distance from the chord to the circle center. This is a simple application of Pythagoras' Theorem.
$Length = 2\sqrt{r^2 - d^2}$, where r is the radius of the circle, d is the perpendicular distance from the chord to the circle center.
• In a circle, a radius perpendicular to a chord bisects the chord. *Converse:* In a circle, a radius that bisects a chord is perpendicular to the chord, *or* In a circle, the perpendicular bisector of a chord passes through the center of the circle.

Angles in a circle

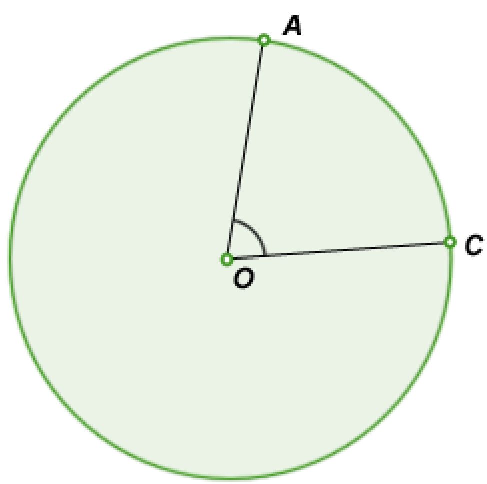
An ***inscribed angle*** is an angle ABC formed by points A, B, and C on the circle's circumference.



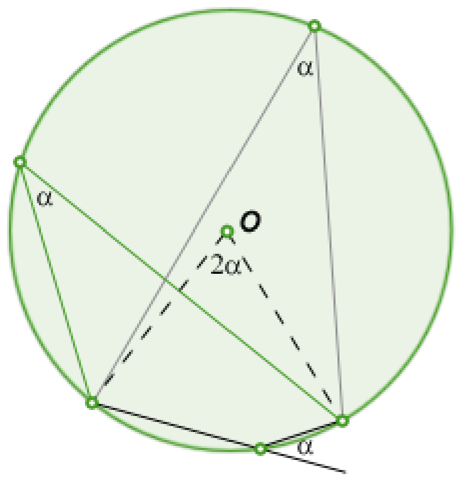
• Given two points A and C, lines from them to a third point B form the inscribed angle ∠ABC. ***Notice*** that the inscribed angle is constant. It only depends on the position of A and C.

• If you know the length L of the minor arc and radius, the inscribed angle is: $Angle = \frac{90L}{\pi r}$

A ***central angle*** is an angle AOC with endpoints A and C located on a circle's circumference and vertex O located at the circle's center. A central angle in a circle determines an arc AC.



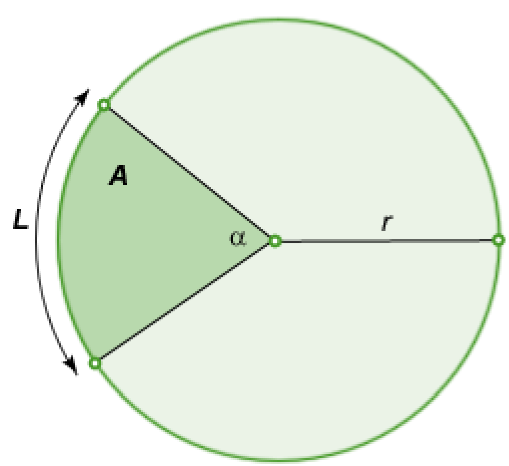
• ***The Central Angle Theorem*** states that the measure of inscribed angle is always half the measure of the central angle.



• An inscribed angle is exactly half the corresponding central angle. Hence, all inscribed angles that subtend the same arc are equal. Angles inscribed on the arc are supplementary. In particular, every inscribed angle that subtends a diameter is a right angle (since the central angle is 180 degrees).

Arcs and Sectors

A portion of the circumference of a circle.



• ***Major and Minor Arcs*** Given two points on a circle, the minor arc is the shortest arc linking them. The major arc is the longest. On the GMAT, we usually assume the minor (shortest) arc.

• ***Arc Length*** The formula the arc measure is: $L = 2\pi r \frac{C}{360}$, where C is the central angle of the arc in degrees. Recall that $2\pi r$ is the circumference of the whole circle, so the formula simply reduces this by the ratio of the arc angle to a full angle (360). By transposing the above formula, you solve for the radius, central angle, or arc length if you know any two of them.

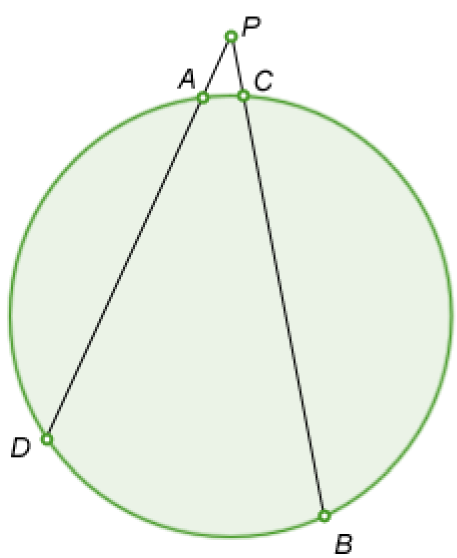
• ***Sector*** is the area enclosed by two radii of a circle and their intercepted arc. A pie-shaped part of a circle.

• ***Area of a sector*** is given by the formula: $Area = \pi r^2 \frac{C}{360}$, where: C is the central angle in degrees. What this formula is doing is taking the area of the whole circle, and then taking a fraction of that depending on the central angle of the sector. So for example, if the central angle was 90°, then the sector would have an area equal to one quarter of the whole circle.

Power of a Point Theorem

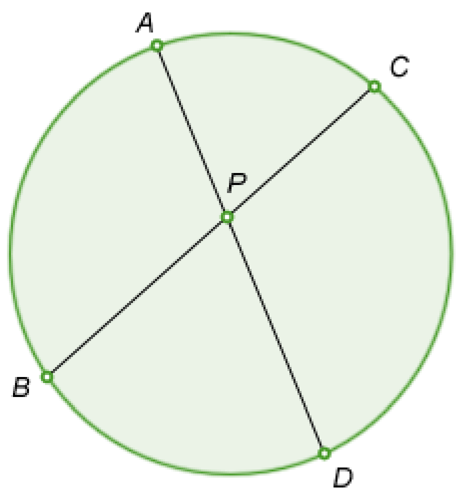
Given circle O, point P *not on the circle*, and a line through P intersecting the circle in two points. The product of the length from P to the first point of intersection and the length from P to the second point of intersection is constant for any choice of a line through P that intersects the circle. This constant is called the ***"power of point P"***.

If P is outside the circle:



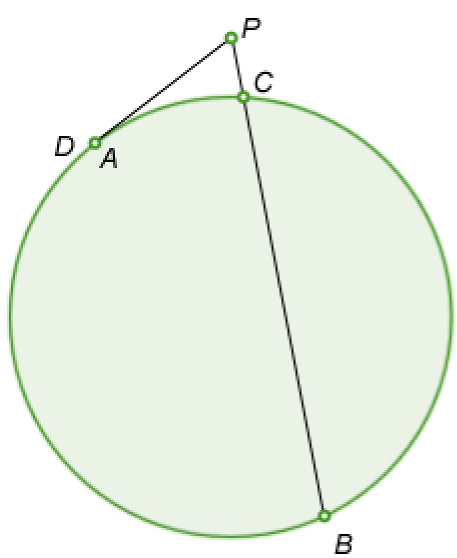
$PA * PD = PC * PB = Constant$ - This becomes the theorem we know as the theorem of intersecting secants.

If P is inside the circle:



$PA * PD = PC * PB = Constant$ - This becomes the theorem we know as the theorem of intersecting chords.

Tangent-Secant

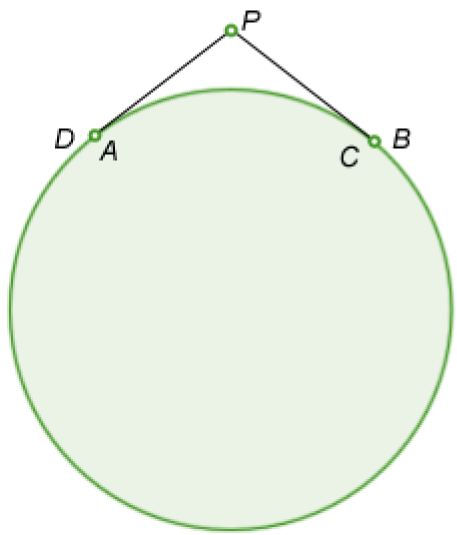


Should one of the lines be tangent to the circle, point A will coincide with point D, and the theorem still applies:

$PA * PD = PC * PB = Constant$

$PA^2 = PC * PB = Constant$ - This becomes the theorem we know as the theorem of secant-tangent theorem.

Two tangents



Should both of the lines be tangents to the circle, point A coincides with point D, point C coincides with point B, and the theorem still applies:

$$PA*PD = PC*PB = Constant$$

$$PA^2 = PC^2$$

$$PA = PC$$

Practice from the GMAT Official Guide:

The Official Guide, 12th Edition: DT #36; PS #33; PS #160; PS #197; PS #212; DS #42; DS #96; DS #114; DS #117; DS #160; DS #173;
The Official Guide, Quantitative 2th Edition: PS #33; PS #141; PS #145; PS #153; PS #162; DS #22; DS #58; DS #59; DS #95; DS #99;
The Official Guide, 11th Edition: DT #36; PS #30; PS #42; PS #100; PS #160; PS #206; PS #229; DS #23; DS #76; DS #86; DS #136; DS #152.

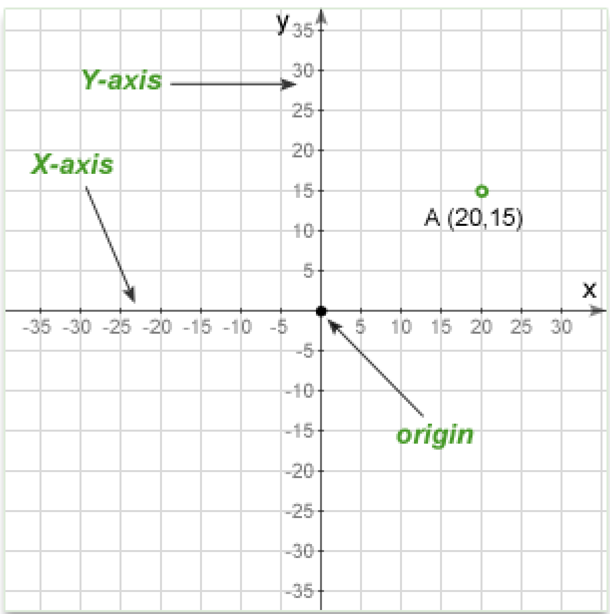
Coordinate Geometry

Definition

Coordinate geometry, or *Cartesian geometry*, is the study of geometry using a coordinate system and the principles of algebra and analysis.

The Coordinate Plane

In coordinate geometry, points are placed on the "coordinate plane" as shown below. The coordinate plane is a two-dimensional surface on which we can plot points, lines and curves. It has two scales, called the x-axis and y-axis, at right angles to each other. The plural of axis is 'axes' (pronounced "AXE-ease").



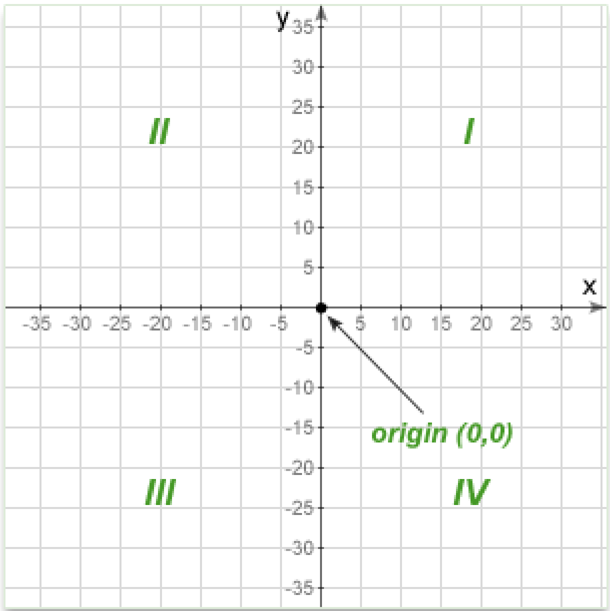
A point's location on the plane is given by two numbers, one that tells where it is on the x-axis and another which tells where it is on the y-axis. Together, they define a single, unique position on the plane. So in the diagram above, the point A has an x value of 20 and a y value of 15. These are the coordinates of the point A, sometimes referred to as its "rectangular coordinates".

X axis
The horizontal scale is called the x-axis and is usually drawn with the zero point in the middle. Values to the right are positive and those to the left are negative.

Y axis
The vertical scale is called the y-axis and is also usually drawn with the zero point in the middle. Values above the origin are positive and those below are negative.

Origin
The point where the two axes cross (at zero on both scales) is called the origin.

Quadrants
When the origin is in the center of the plane, they divide it into four areas called quadrants.



The first quadrant, by convention, is the top right, and then they go around counter-clockwise. In the diagram above they are labeled Quadrant 1, 2 etc. It is conventional to label them with numerals but we talk about them as "first, second, third, and fourth quadrant".

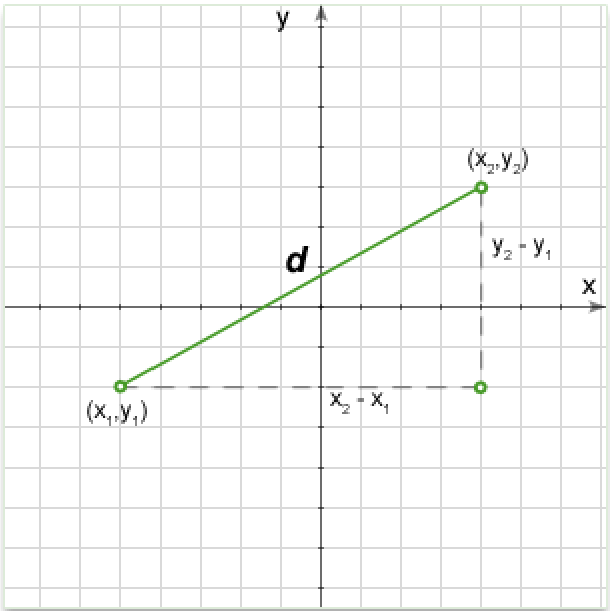
Point (x,y)
The coordinates are written as an "ordered pair". The letter P is simply the name of the point and is used to distinguish it from others.

The two numbers in parentheses are the x and y coordinates of the point. The first number (*x*) specifies how far along the x (horizontal) axis the point is. The second is the *y* coordinate and specifies how far up or down the y axis to go. It is called an ordered pair because the order of the two numbers matters - the first is always the x (horizontal) coordinate.

The sign of the coordinate is important. A positive number means to go to the right (x) or up (y). Negative numbers mean to go left (x) or down (y).

Distance between two points

Given coordinates of two points, distance D between two points is given by:
$D=\sqrt{dx^2+dy^2}$ (where dx is the difference between the x-coordinates and dy is the difference between the y-coordinates of the points)



As you can see, the distance formula on the plane is derived from the Pythagorean theorem.

Above formula can be written in the following way for given two points (x_1,y_1) and (x_2,y_2):

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Vertical and horizontal lines
If the line segment is exactly vertical or horizontal, the formula above will still work fine, but there is an easier way. For a horizontal line, its length is the difference between the x-coordinates. For a vertical line its length is the difference between the y-coordinates.

Distance between the point A (x,y) and the origin
As the one point is origin with coordinate O (0,0) the formula can be simplified to:

$$D=\sqrt{x^2+y^2}$$

Example #1
Q: Find the distance between the point A (3,-1) and B (-1,2)
Solution: Substituting values in the equation we'll get

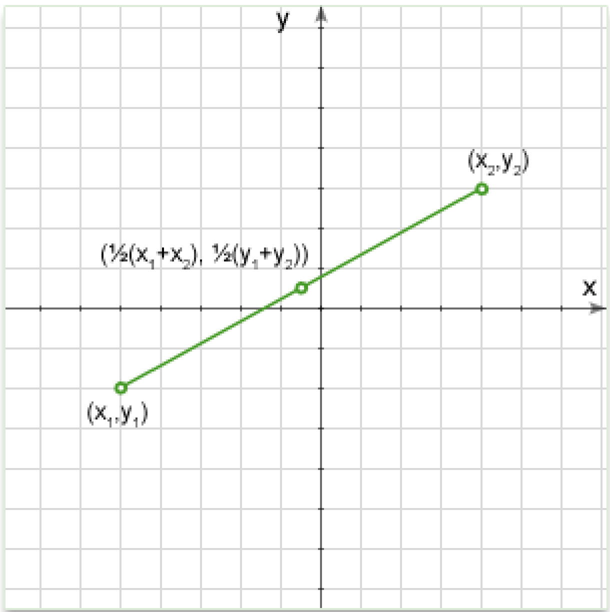
$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$D=\sqrt{(-1-3)^2+(2-(-1))^2}=\sqrt{16+9}=5$$

Midpoint of a Line Segment

A line segment on the coordinate plane is defined by two endpoints whose coordinates are known. The midpoint of this line is exactly halfway between these endpoints and its location can be found using the Midpoint Theorem, which states:

• The x-coordinate of the midpoint is the average of the x-coordinates of the two endpoints.
• Likewise, the y-coordinate is the average of the y-coordinates of the endpoints.

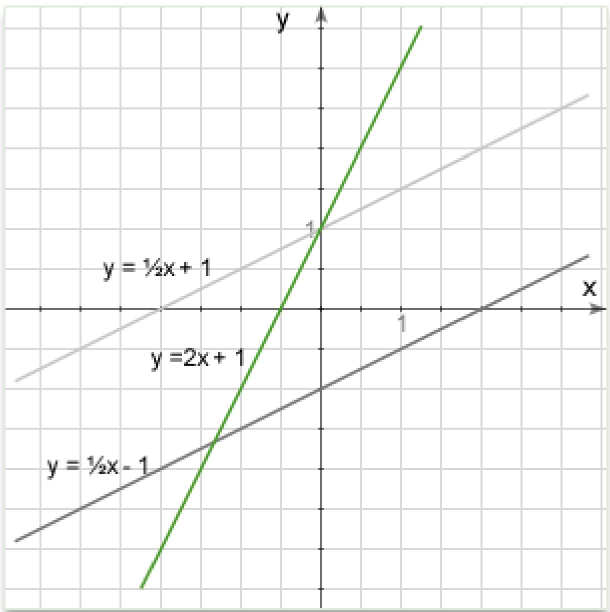


Coordinates of the midpoint $M(x_m,y_m)$ of the line segment AB, ($A(x_1,y_1)$ and $B(x_2,y_2)$) are $x_m=\frac{x_1+x_2}{2}$ and $y_m=\frac{y_1+y_2}{2}$

Lines in Coordinate Geometry

In Euclidean geometry, a line is a straight curve. In coordinate geometry, lines in a Cartesian plane can be described algebraically by linear equations and linear functions.

Every straight line in the plane can represented by a first degree equation with two variables.

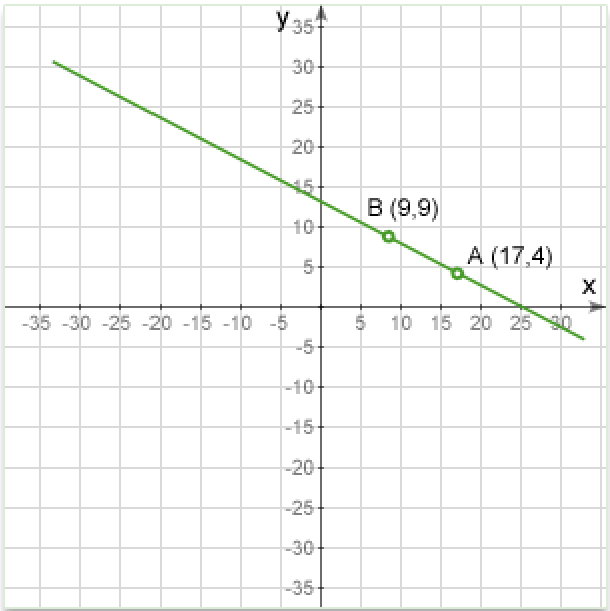


There are several approaches commonly used in coordinate geometry. It does not matter whether we are talking about a line, ray or line segment. **In all cases any of the below methods will provide enough information to define the line exactly.**

1. General form.
The general form of the equation of a straight line is
$ax+by+c=0$
Where a, b and c are arbitrary constants. This form includes all other forms as special cases. For an equation in this form the slope is $-\frac{a}{b}$ and the y intercept is $-\frac{c}{b}$.

2. Point-intercept form.
$y=mx+b$
Where: m is the slope of the line; b is the y-intercept of the line; x is the independent variable of the function y.

3. Using two points
In figure below, a line is defined by the two points A and B. By providing the coordinates of the two points, we can draw a line. No other line could pass through both these points and so the line they define is unique.



The equation of a straight line passing through points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ is:

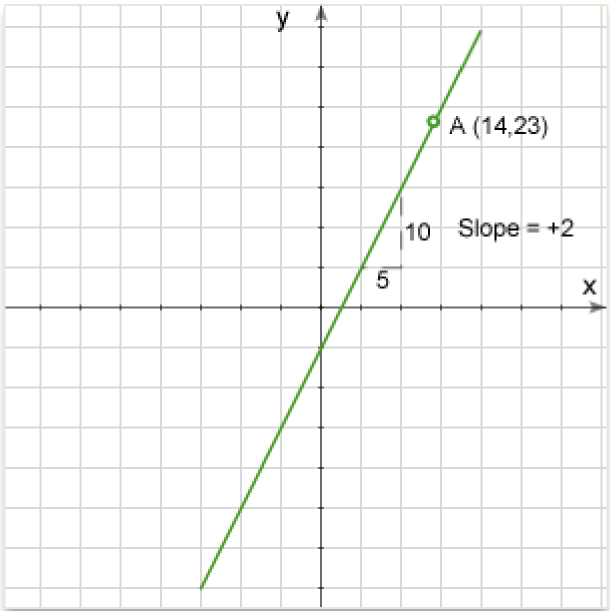
$$\frac{y-y_1}{x-x_1}=\frac{y_1-y_2}{x_1-x_2}$$

Example #1
Q: Find the equation of a line passing through the points **A (17,4)** and **B (9,9)**.
Solution: Substituting the values in equation $\frac{y-y_1}{x-x_1}=\frac{y_1-y_2}{x_1-x_2}$ we'll get: $\frac{y-4}{x-17}=\frac{4-9}{17-9}$

$\frac{y-4}{x-17}=\frac{-5}{8}$ --> $8y-32=-5x+85$ --> $8y+5x-117=0$ OR if we want to write the equation in the slope-intercept form: $y=-\frac{5}{8}x+\frac{117}{8}$

4. Using one point and the slope
Sometimes on the GMAT you will be given a point on the line and its slope and from this information you will need to find the equation or check if this line goes through another point. You can think of the slope as the direction of the line. So once you know that a line goes through a certain point, and which direction it is pointing, you have defined one unique line.

In figure below, we see a line passing through the point A at (14,23). We also see that it's slope is +2 (which means it goes 2 up for every one across). With these two facts we can establish a unique line.



The equation of a straight line that passes through a point $P_1(x_1,y_1)$ with a slope m is:

$$y-y_1=m(x-x_1)$$

Example #2
Q: Find the equation of a line passing through the point A (14,23) and the slope 2.
Solution: Substituting the values in equation $y-y_1=m(x-x_1)$ we'll get $y-23=2(x-14)$ --> $y=2x-5$

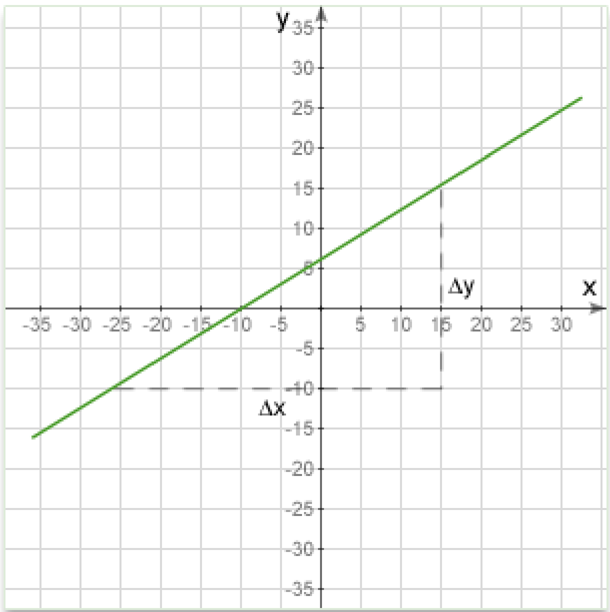
4. Intercept form.
The equation of a straight line whose x and y intercepts are a and b, respectively, is:
$$\frac{x}{a}+\frac{y}{b}=1$$

Example #3
Q: Find the equation of a line whose x intercept is 5 and y intercept is 2.
Solution: Substituting the values in equation $\frac{x}{a}+\frac{y}{b}=1$ we'll get $\frac{x}{5}+\frac{y}{2}=1$ --> $5y+2x-10=0$ OR if we want to write the equation in the slope-intercept form: $y=-\frac{2}{5}x+2$

Slope of a Line

The slope or gradient of a line describes its steepness, incline, or grade. **A higher slope value indicates a steeper incline.**

The slope is defined as the ratio of the "rise" divided by the "run" between two points on a line, or in other words, the ratio of the altitude change to the horizontal distance between any two points on the line.



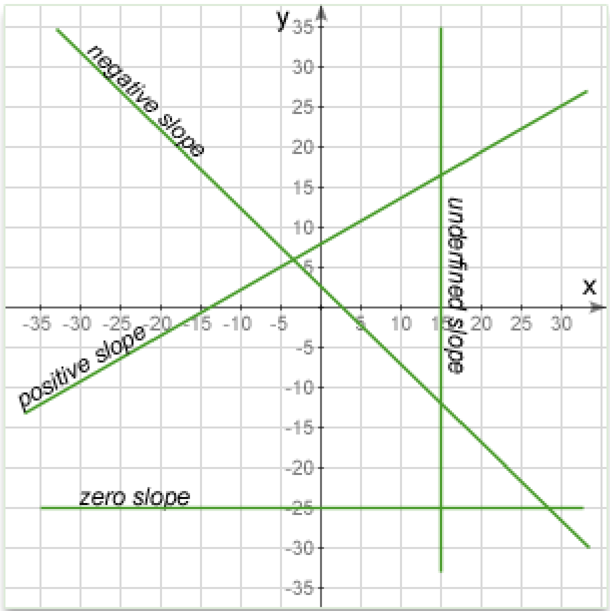
Given two points (x_1,y_1) and (x_2,y_2) on a line, the slope m of the line is:

$$m=\frac{y_2-y_1}{x_2-x_1}$$

If the equation of the line is given in the **Point-intercept form**: $y=mx+b$, then m is the slope. This form of a line's equation is called the slope-intercept form, because b can be interpreted as the y-intercept of the line, the y-coordinate where the line intersects the y-axis.

If the equation of the line is given in the **General form:** $ax+by+c=0$, then the slope is $-\frac{a}{b}$ and the y intercept is $-\frac{c}{b}$.

SLOPE DIRECTION
The slope of a line can be positive, negative, zero or undefined.



Positive slope
Here, y increases as x increases, so the line slopes upwards to the right. The slope will be a positive number. The line below has a slope of about +0.3, it goes up about 0.3 for every step of 1 along the x-axis.

Negative slope
Here, y decreases as x increases, so the line slopes downwards to the right. The slope will be a negative number. The line below has a slope of about -0.3, it goes down about 0.3 for every step of 1 along the x-axis.

Zero slope
Here, y does not change as x increases, so the line in exactly horizontal. The slope of any horizontal line is always zero. The line below goes neither up nor down as x increases, so its slope is zero.
Undefined slope
When the line is exactly vertical, it does not have a defined slope. The two x coordinates are the same, so the difference is zero. The slope calculation is then something like $slope = \frac{15}{0}$ When you divide anything by zero the result has no meaning. The line above is exactly vertical, so it has no defined slope.

SLOPE AND QUADRANTS:

1. **If the slope of a line is negative**, the line WILL intersect quadrants II and IV. X and Y intersects of the line with negative slope have the same sign. Therefore if X and Y intersects are positive, the line intersects quadrant I; if negative, quadrant III.

2. **If the slope of line is positive**, line WILL intersect quadrants I and III. Y and X intersects of the line with positive slope have opposite signs. Therefore if X intersect is negative, line intersects the quadrant II too, if positive quadrant IV.

3. **Every line (but the one crosses origin OR parallel to X or Y axis OR X and Y axis themselves) crosses three quadrants**. Only the line which crosses origin $(0,0)$ OR is parallel to either of axis crosses only two quadrants.

4. **If a line is horizontal** it has a slope of 0, is parallel to X-axis and crosses quadrant I and II if the Y intersect is positive OR quadrants III and IV, if the Y intersect is negative. Equation of such line is y=b, where *b* is y intersect.

5. **If a line is vertical**, the slope is not defined, line is parallel to Y-axis and crosses quadrant I and IV, if the X intersect is positive and quadrant II and III, if the X intersect is negative. Equation of such line is $x = a$, where *a* is x-intercept.

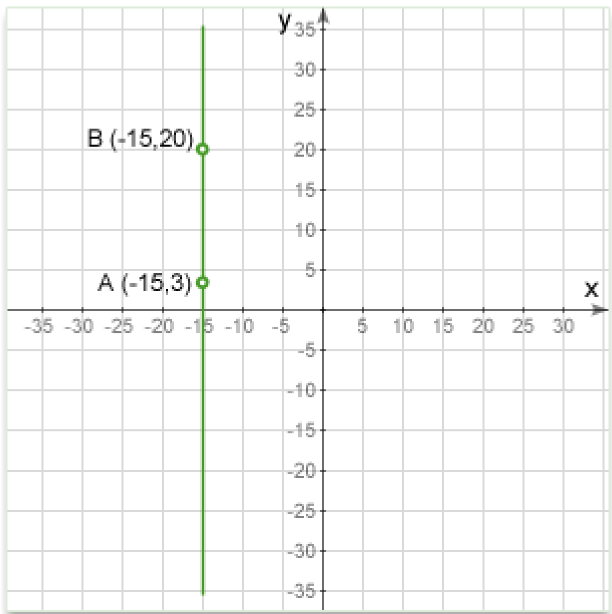
6. **For a line that crosses two points** (x_1,y_1) **and** (x_2,y_2), slope $m=\frac{y_2-y_1}{x_2-x_1}$

7. **If the slope is 1** the angle formed by the line is 45 degrees.

8. **Given a point and slope, equation of a line can be found**. The equation of a straight line that passes through a point (x_1,y_1) with a slope m is: $y-y_1=m(x-x_1)$

Vertical and horizontal lines

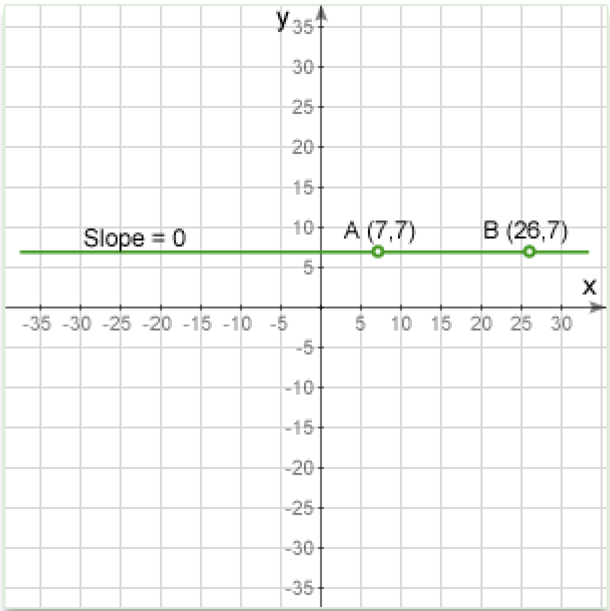
A vertical line is parallel to the y-axis of the coordinate plane. All points on the line will have the same x-coordinate.



A vertical line has no slope. Or put another way, for a vertical line the slope is undefined.

The equation of a vertical line is:
$x=a$
Where: x is the coordinate of any point on the line; a is where the line crosses the x-axis (x intercept). Notice that the equation is independent of y. Any point on the vertical line satisfies the equation.

A horizontal line is parallel to the x-axis of the coordinate plane. All points on the line will have the same y-coordinate.

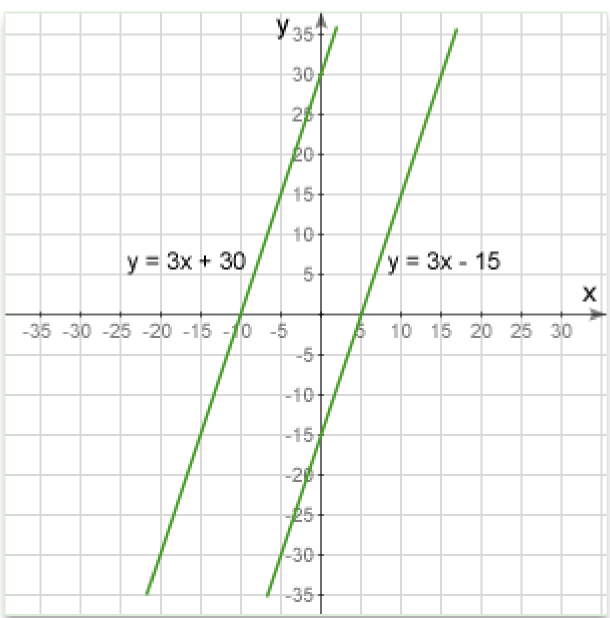


A horizontal line has a slope of zero.

The equation of a horizontal line is:
$y = b$
Where: x is the coordinate of any point on the line; b is where the line crosses the y-axis (y intercept). Notice that the equation is independent of x. Any point on the horizontal line satisfies the equation.

Parallel lines

Parallel lines have the same slope.



The slope can be found using any method that is convenient to you:

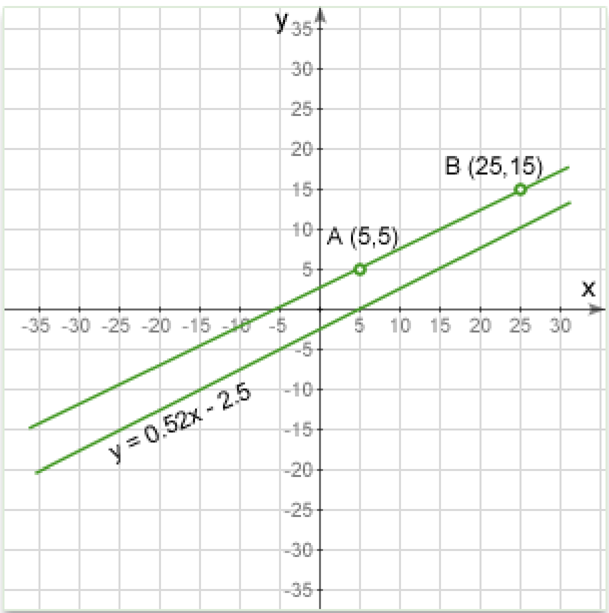
From two given points on the line.
From the equation of the line in slope-intercept form
From the equation of the line in point-slope form

The equation of a line through the point $P_1(x_1,y_1)$ **and parallel to line** $ax+by+c=0$ **is:**
$a(x-x_1)+b(y-y_1)=0$

Distance between two parallel lines $y=mx+b$ and $y=mx+c$ can be found by the formula:

$$D=\frac{|b-c|}{\sqrt{m^2+1}}$$

Example #1
Q:There are two lines. One line is defined by two points at (5,5) and (25,15). The other is defined by an equation in slope-intercept form y = 0.52x - 2.5. Are two lines parallel?

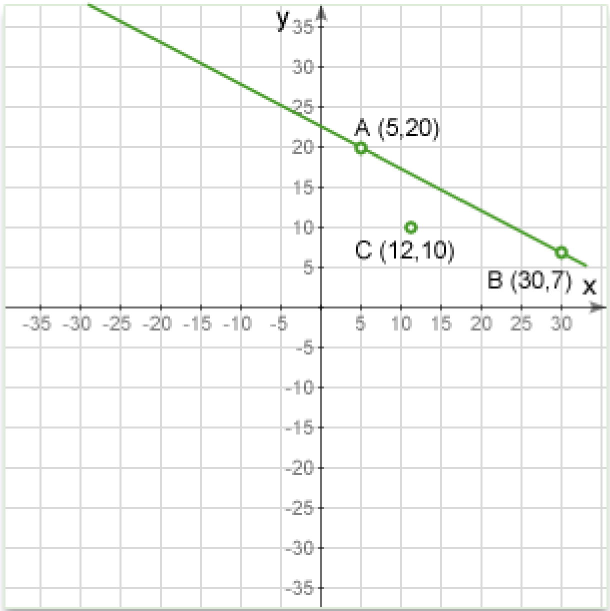


Solution:
For the top line, the slope is found using the coordinates of the two points that define the line. $Slope=\frac{15-5}{25-5}=0.5$

For the lower line, the slope is taken directly from the formula. Recall that the slope intercept formula is y = mx + b, where m is the slope. So looking at the formula we see that the slope is 0.52.

So, the top one has a slope of 0.5, the lower slope is 0.52, which are not equal. Therefore, the lines are not parallel.

Example #2
Q: Define a line through a point C parallel to a line passes through the points A and B.



Solution: We first find the slope of the line AB using the same method as in the example above.

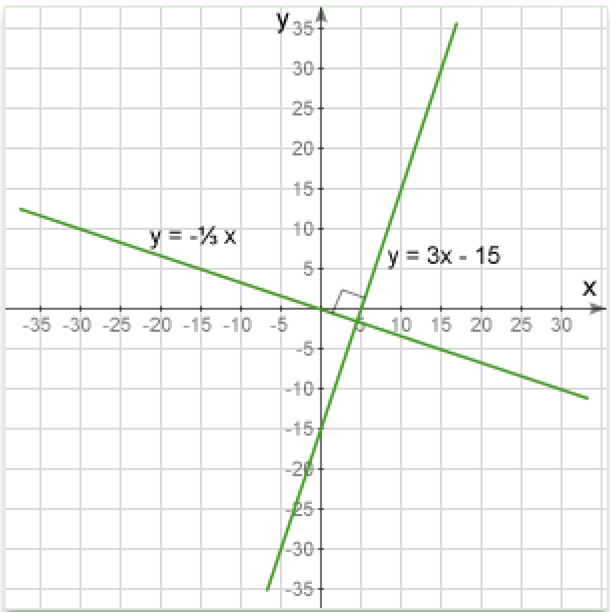
$$Slope AB = \frac{20-7}{5-30} = -0.52$$

For the line to be parallel to AB it will have the same slope, and will pass through a given point, C(12,10). We therefore have enough information to define the line by its equation in point-slope form:

$y = -0.52(x-12)+10$ --> $y = -0.52x+16.24$

Perpendicular lines

For one line to be perpendicular to another, the relationship between their slopes has to be **negative reciprocal** $-\frac{1}{m}$. In other words, the two lines are perpendicular if and only if the product of their slopes is -1.

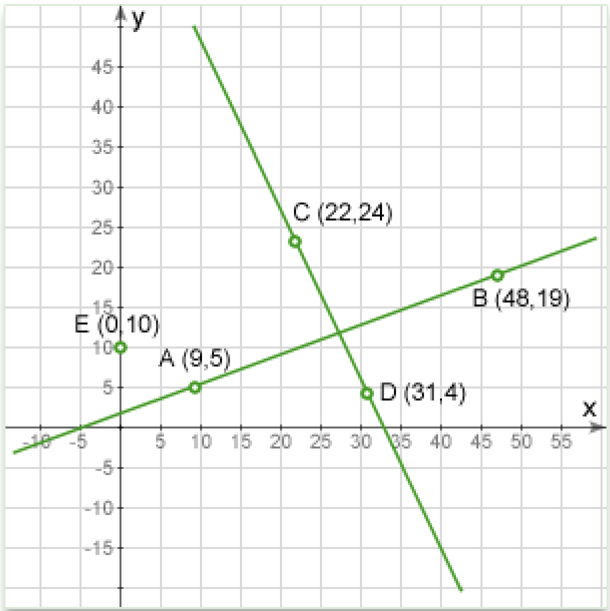


The two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are perpendicular if $a_1a_2+b_1b_2=0$.

The equation of a line passing through the point $P_1(x_1,y_1)$ and perpendicular to line $ax+by+c=0$ is: $b(x-x_1)-a(y-y_1)=0$

Example #1
Q: Are the two lines below perpendicular?

Solution:



To answer, we must find the slope of each line and then check to see if one slope is the negative reciprocal of the other or if their product equals to -1.

$$SlopeAB = \frac{5-19}{9-48} = \frac{-14}{-39} = 0.358$$

$$SlopeCD = \frac{24-4}{22-31} = \frac{20}{-9} = -2.22$$

If the lines are perpendicular, each will be the negative reciprocal of the other. It doesn't matter which line we start with, so we will pick AB:

Negative reciprocal of 0.358 is $-\frac{1}{0.358} = -2.79$

So, the slope of CD is -2.22, and the negative reciprocal of the slope of AB is -2.79. These are not the same, so the lines are not perpendicular, even though they may look as though they are. However, if you looked carefully at the diagram, you might have noticed that point C is a little too far to the left for the lines to be perpendicular.

Example #2

Q: Define a line passing through the point E and perpendicular to a line passing through the points C and D on the graph above.

Solution: The point E is on the y-axis and so is the y-intercept of the desired line. Once we know the slope of the line, we can express it using its equation in slope-intercept form y=mx+b, where m is the slope and b is the y-intercept.

First find the slope of line CD:

$$SlopeCD = \frac{24-4}{22-31} = \frac{20}{-9} = -2.22$$

The line we seek will have a slope which is the negative reciprocal of:

$$-\frac{1}{-2.22} = 0.45$$

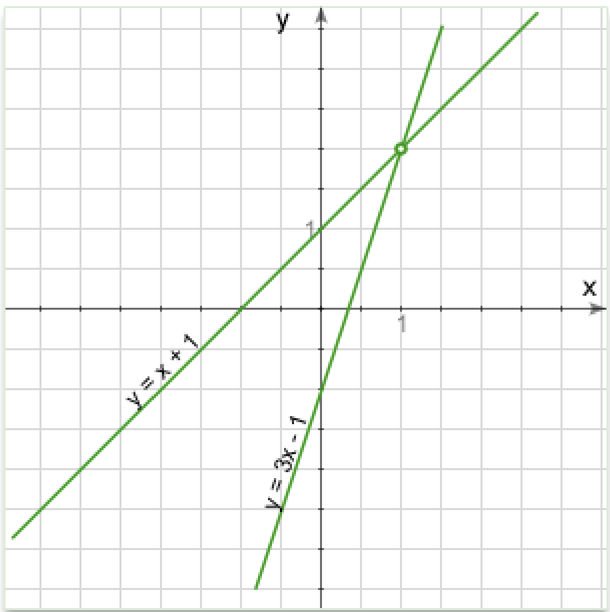
Since E is on the Y-axis, we know that the intercept is 10. Plugging these values into the line equation, the line we need is described by the equation

$$y = 0.45x + 10$$

This is one of the ways a line can be defined and so we have solved the problem. If we wanted to plot the line, we would find another point on the line using the equation and then draw the line through that point and the intercept.

Intersection of two straight lines

The point of intersection of two non-parallel lines can be found from the equations of the two lines.



To find the intersection of two straight lines:

1. First we need their equations
2. Then, since at the point of intersection, the two equations will share a point and thus have the same values of x and y, we set the two equations equal to each other. This gives an equation that we can solve for x
3. We substitute the x value in one of the line equations (it doesn't matter which) and solve it for y.

This gives us the x and y coordinates of the intersection.

Example #1
Q: Find the point of intersection of two lines that have the following equations (in slope-intercept form):
$y = 3x - 3$

$y = 2.3x + 4$

Solution: At the point of intersection they will both have the same y-coordinate value, so we set the equations equal to each other:
$3x - 3 = 2.3x + 4$

This gives us an equation in one unknown (x) which we can solve:
$x = 10$

To find y, simply set x equal to 10 in the equation of either line and solve for y:
Equation for a line $y = 3x - 3$ (Either line will do)
Set x equal to 10: $y = 30 - 3$
$y = 27$

We now have both x and y, so the intersection point is (10, 27)

Example #2
Q: Find the point of intersection of two lines that have the following equations (in slope-intercept form): $y = 3x - 3$ and $x = 12$ (A vertical line)
Solution: When one of the lines is vertical, it has no defined slope. We find the intersection slightly differently.

On the vertical line, all points on it have an x-coordinate of 12 (the definition of a vertical line), so we simply set x equal to 12 in the first equation and solve it for y.
Equation for a line $y = 3x - 3$
Set x equal to 12 $y = 36 - 3$
$y = 33$
So the intersection point is at (12,33).

Note: If both lines are vertical or horizontal, they are parallel and have no intersection

Distance from a point to a line

The distance from a point to a line is the **shortest** distance between them - the length of a perpendicular line segment from the line to the point.

The distance from a point (x_0, y_0) to a line $ax + by + c = 0$ is given by the formula:

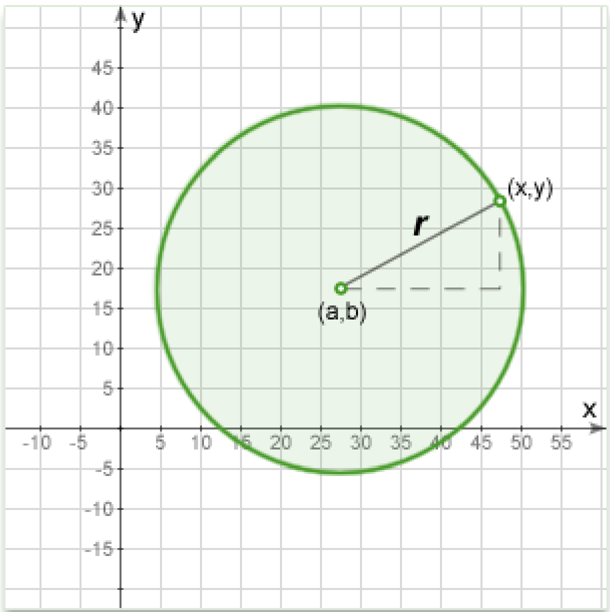
$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

- **When the line is horizontal the formula transforms to:** $D = |P_y - L_y|$
  Where: P_y is the y-coordinate of the given point P; L_y is the y-coordinate of **any** point on the given vertical line L. | | the vertical bars mean "absolute value" - make it positive even if it calculates to a negative.
- **When the line is vertical the formula transforms to:** $D = |P_x - L_x|$
  Where: P_x is the x-coordinate of the given point P; L_x is the x-coordinate of **any** point on the given vertical line L. | | the vertical bars mean "absolute value" - make it positive even if it calculates to a negative.
- **When the given point is origin**, then the distance between origin and line ax+by+c=0 is given by the formula:
  $$D = \frac{|c|}{\sqrt{a^2 + b^2}}$$

Circle on a plane

In an x-y Cartesian coordinate system, the circle with center (a, b) and radius r is the set of all points (x, y) such that:

$$(x-a)^2+(y-b)^2=r^2$$



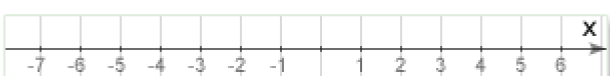
This equation of the circle follows from the Pythagorean theorem applied to any point on the circle: as shown in the diagram above, the radius is the hypotenuse of a right-angled triangle whose other sides are of length x-a and y-b.

If the circle is centered at the origin (0, 0), then the equation simplifies to:

$$x^2+y^2=r^2$$

Number line

A number line is a picture of a straight line on which every point corresponds to a real number and every real number to a point.



On the GMAT we can often see such statement: k is halfway between m and n on the number line. Remember this statement can ALWAYS be expressed as:

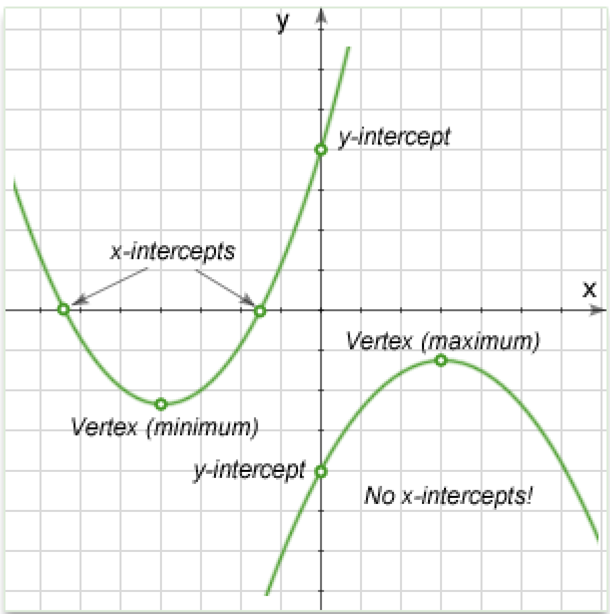
$$\frac{m+n}{2}=k.$$

Also on the GMAT we can often see another statement: The distance between p and m on the number line is the same as the distance between p and n. Remember this statement can ALWAYS be expressed as:

$$|p-m|=|p-n|.$$

Parabola

A parabola is the graph associated with a quadratic function, i.e. a function of the form $y=ax^2+bx+c$.



The general or standard form of a quadratic function is $y=ax^2+bx+c$, or in function form, $f(x)=ax^2+bx+c$, where x is the independent variable, y is the dependent variable, and a, b, and c are constants.

- The larger the absolute value of a, the steeper (or thinner) the parabola is, since the value of y is increased more quickly.
- If a is positive, the parabola opens upward, if negative, the parabola opens downward.

x-intercepts: The x-intercepts, **if any**, are also called the roots of the function. The x-intercepts are the solutions to the equation $0=ax^2+bx+c$ and can be calculated by the formula:

$x_1=\frac{-b-\sqrt{b^2-4ac}}{2a}$ and $x_2=\frac{-b+\sqrt{b^2-4ac}}{2a}$

Expression b^2-4ac is called *discriminant*:

- If discriminant is positive parabola has two intercepts with x-axis;
- If discriminant is negative parabola has no intercepts with x-axis;
- If discriminant is zero parabola has one intercept with x-axis (tangent point).

y-intercept: Given $y=ax^2+bx+c$, the y-intercept is c, as y intercept means the value of y when x=0.

Vertex: The vertex represents the maximum (or minimum) value of the function, and is very important in calculus.

The vertex of the parabola is located at point $\left(-\frac{b}{2a}, c-\frac{b^2}{4a}\right)$.

Note: typically just $-\frac{b}{2a}$, is calculated and plugged in for x to find y.

Practice from the GMAT Official Guide:

The Official Guide, 12th Edition: DT #39; PS #9; PS #25; PS #39; PS #88; PS #194; PS #205; PS #210; PS #212; PS #229; DS #69; DS #75; DS #93; DS #94; DS #108; DS #121; DS #149; DS #164;
The Official Guide, Quantitative 2th Edition: PS #21; PS #85; PS #102; PS #123; DS #22;
The Official Guide, 11th Edition: DT #39; PS #7; PS #23; PS #36; PS #89; PS #199; PS #222; PS #227; PS #229; PS #248; DS #15; DS #78; DS #85; DS #124

Standard Deviation

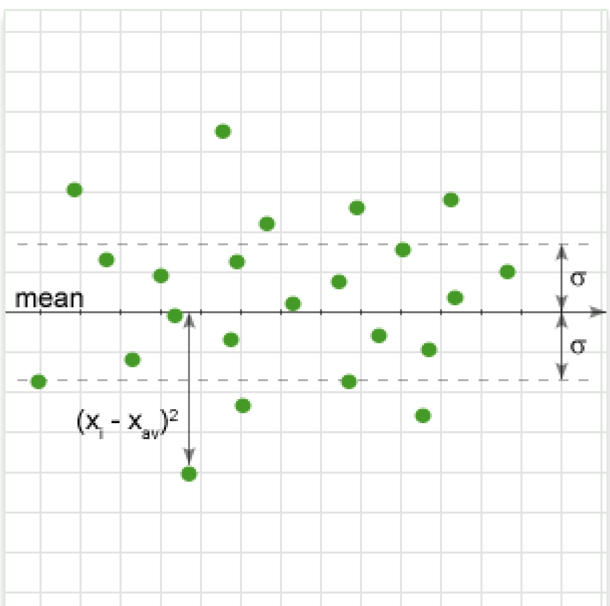
Definition

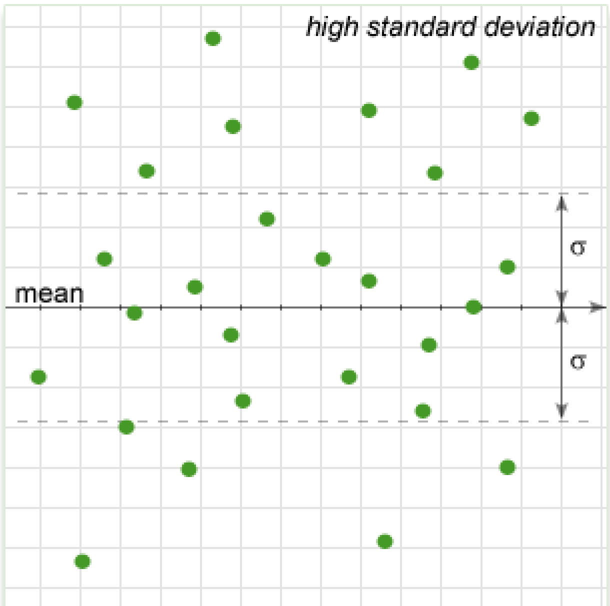
Standard Deviation (SD, or STD or σ) - a measure of the dispersion or variation in a distribution, equal to the square root of variance or the arithmetic mean (average) of squares of deviations from the arithmetic mean.

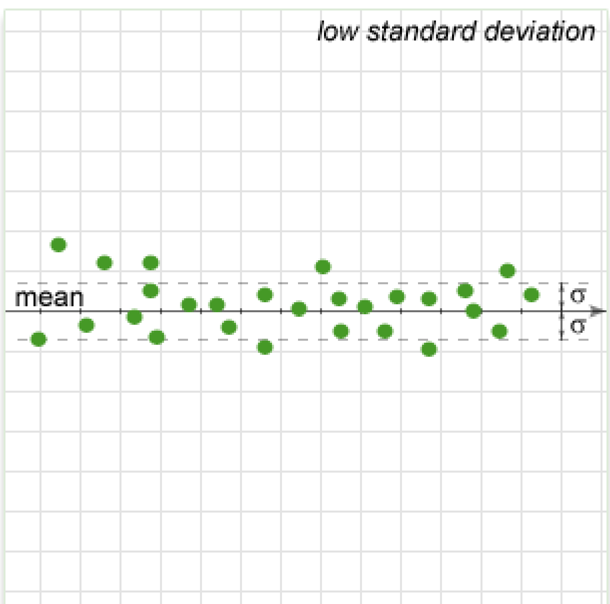
$$variance = \frac{\sum (x_i - x_{av})^2}{N}$$

$$\sigma = \sqrt{variance} = \sqrt{\frac{\sum (x_i - x_{av})^2}{N}}$$

In simple terms, it shows how much variation there is from the "average" (mean). It may be thought of as the average difference from the mean of distribution, how far data points are away from the mean. A low standard deviation indicates that data points tend to be very close to the mean, whereas high standard deviation indicates that the data are spread out over a large range of values.







Properties

$\sigma \geq 0$;

$\sigma = 0$ only if <u>all</u> elements in a set is equal;

Let standard deviation of $\{x_i\}$ be σ and mean of the set be μ:

Standard deviation of $\left\{\frac{x_i}{a}\right\}$ is $\sigma' = \frac{\sigma}{a}$. Decrease/increase in all elements of a set by a constant percentage will decrease/increase standard deviation of the set by the same percentage.

Standard deviation of $\{x_i + a\}$ is $\sigma' = \sigma$. Decrease/increase in all elements of a set by a constant value DOES NOT decrease/increase standard deviation of the set.

if a new element y is added to $\{x_i\}$ set and standard deviation of a new set $\{\{x_i\}, y\}$ is σ', then:

1) $\sigma' > \sigma$ if $|y - \mu| > \sigma$

2) $\sigma' = \sigma$ if $|y - \mu| = \sigma$

3) $\sigma' < \sigma$ if $|y - \mu| < \sigma$

4) σ' is the lowest if $y = \mu$

Tips and Tricks

GMAC in majority of problems doesn't ask you to calculate standard deviation. Instead it tests your intuitive understanding of the concept. In 90% cases it is a faster way to use just average of $|x_i - x_{av}|$ instead of true formula for standard deviation, and treat standard deviation as "average difference between elements and mean". Therefore, before trying to calculate standard deviation, maybe you can solve a problem much faster by using just your intuition.

Advance tip. Not all points contribute equally to standard deviation. Taking into account that standard deviation uses sum of squares of deviations from mean, the most remote points will essentially contribute to standard deviation. For example, we have a set A that has a mean of 5. The point 10 gives $(10-5)^2 = 25$ in sum of squares but point 6 gives only $(6-5)^2 = 1$. 25 times the difference! So, when you need to find what set has the largest standard deviation, always look for set with the largest range because remote points have a very significant contribution to standard deviation.

Examples

Example #1
Q: There is a set $\{67, 32, 76, 35, 101, 45, 24, 37\}$. If we create a new set that consists of all elements of the initial set but decreased by 17%, what is the change in standard deviation?
Solution: We don't need to calculate as we know rule that decrease in all elements of a set by a constant percentage will decrease standard deviation of the set by the same percentage. So, the decrease in standard deviation is 17%.

Example #2
Q: There is a set of consecutive even integers. What is the standard deviation of the set?
(1) There are 39 elements in the set.
(2) the mean of the set is 382.
Solution: Before reading Data Sufficiency statements, what can we say about the question? What should we know to find standard deviation? "consecutive even integers" means that all elements strictly related to each other. If we shift the set by adding or subtracting any integer, does it change standard deviation (average deviation of elements from the mean)? No. One thing we should know is the number of elements in the set, because the more elements we have the broader they are distributed relative to the mean. Now, look at DS statements, all we need it is just first statement. So, A is sufficient.

Example #3
Q: Standard deviation of set $\{23, 31, 76, 45, 16, 55, 54, 36\}$ is 18.3. How many elements are 1 standard deviation above the mean?

Solution: Let's find mean: $\mu = \frac{23+31+76+45+16+55+54+36}{8} = 42$

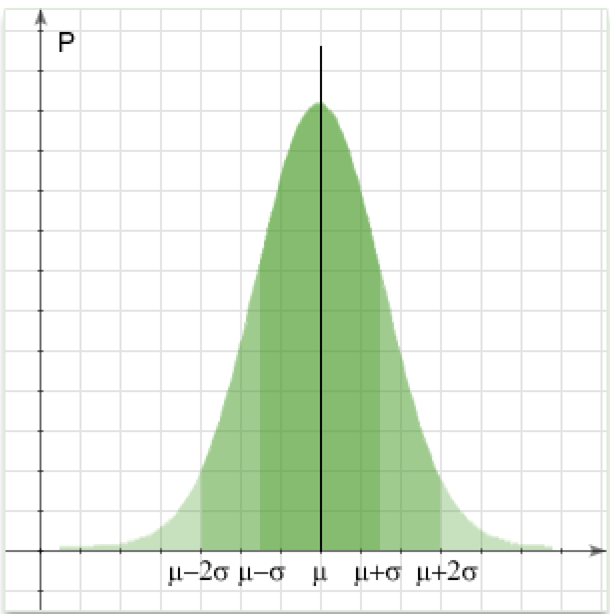
Now, we need to count all numbers greater than 42+18.3=60.3. It is one number - 76. The answer is 1.

Example #4
Q: There is a set A of 19 integers with mean 4 and standard deviation of 3. Now we form a new set B by adding 2 more elements to the set A. What two elements will decrease the standard deviation the most?
A) 9 and 3
B) -3 and 3
C) 6 and 1
D) 4 and 5
E) 5 and 5
Solution: The closer to the mean, the greater decrease in standard deviation. D has 4 (equal our mean) and 5 (differs from mean only by 1). All other options have larger deviation from mean.

Normal distribution

It is a more advance concept that you will never see in GMAT but understanding statistic properties of standard deviation can help you to be more confident about simple properties stated above.

In probability theory and statistics, the normal distribution or Gaussian distribution is a continuous probability distribution that describes data that cluster around a mean or average. Majority of statistical data can be characterized by normal distribution.



$\mu-\sigma<x<\mu+\sigma$ covers 68% of data

$\mu-2\sigma<x<\mu+2\sigma$ covers 95% of data

$\mu-3\sigma<x<\mu+3\sigma$ covers 99% of data

Practice from the GMAT Official Guide:
The Official Guide, 12th Edition: DT #9; DT #31; PS #199; DS #134;
The Official Guide, 11th Edition: DT #31; PS #212;

Resources
Bunuel's post with PS SD-problems: [PS Standard Deviation Problems]
Bunuel's post with DS SD-problems: [DS Standard Deviation Problems]

Probability

Definition

A number expressing the probability (p) that a specific event will occur, expressed as the ratio of the number of actual occurrences (n) to the number of possible occurrences (N).

$$p = \frac{n}{N}$$

A number expressing the probability (q) that a specific event will not occur:

$$q = \frac{(N-n)}{N} = 1-p$$

Examples

Coin

There are two equally possible outcomes when we toss a coin: a head (H) or tail (T). Therefore, the probability of getting head is 50% or $\frac{1}{2}$ and the probability of getting tail is 50% or $\frac{1}{2}$.
All possibilities: {H,T}

Dice

There are 6 equally possible outcomes when we roll a die. The probability of getting any number out of 1-6 is $\frac{1}{6}$.
All possibilities: {1,2,3,4,5,6}

Marbles, Balls, Cards...

Let's assume we have a jar with 10 green and 90 white marbles. If we randomly choose a marble, what is the probability of getting a green marble?
The number of all marbles: N = 10 + 90 =100
The number of green marbles: n = 10

Probability of getting a green marble: $p = \frac{n}{N} = \frac{10}{100} = \frac{1}{10}$

There is one important concept in problems with marbles/cards/balls. When the first marble is removed from a jar and not replaced, the probability for the second marble differs ($\frac{9}{99}$ vs. $\frac{10}{100}$). Whereas in case of a coin or dice the probabilities are always the same ($\frac{1}{6}$and $\frac{1}{2}$). Usually, a problem explicitly states: it is a problem with replacement or without replacement.

Independent events

Two events are independent if occurrence of one event does not influence occurrence of other events. For n independent events the probability is the product of all probabilities of independent events:

p = p1 * p2 * ... * pn-1 * pn

or

P(A and B) = P(A) * P(B) - A and B denote independent events

Example #1
Q:There is a coin and a die. After one flip and one toss, what is the probability of getting heads and a "4"?

Solution: Tossing a coin and rolling a die are independent events. The probability of getting heads is $\frac{1}{2}$ and probability of getting a "4" is $\frac{1}{6}$. Therefore, the probability of getting heads and a "4" is:
$P = \frac{1}{2} * \frac{1}{6} = \frac{1}{12}$

Example #2
Q: If there is a 20% chance of rain, what is the probability that it will rain on the first day but not on the second?
Solution: The probability of rain is 0.2; therefore probability of sunshine is q = 1 - 0.2 = 0.8. This yields that the probability of rain on the first day and sunshine on the second day is:
P = 0.2 * 0.8 = 0.16

Example #3
Q:There are two sets of integers: {1,3,6,7,8} and {3,5,2}. If Robert chooses randomly one integer from the first set and one integer from the second set, what is the probability of getting two odd integers?
Solution: There is a total of 5 integers in the first set and 3 of them are odd: {1, 3, 7}. Therefore, the probability of getting odd integer out of first set is $\frac{3}{5}$. There are 3 integers in the second set and 2 of them are odd: {3, 5}.

Therefore, the probability of getting an odd integer out of second set is $\frac{2}{3}$. Finally, the probability of getting two odd integers is:

$$P=\frac{3}{5}*\frac{2}{3}=\frac{2}{5}$$

Mutually exclusive events

Shakespeare's phrase "To be, or not to be: that is the question" is an example of two mutually exclusive events.

Two events are mutually exclusive if they cannot occur at the same time. For n mutually exclusive events the probability is the sum of all probabilities of events:

p = p1 + p2 + ... + pn-1 + pn

or

P(A or B) = P(A) + P(B) - A and B denotes mutually exclusive events

Example #1
Q: If Jessica rolls a die, what is the probability of getting at least a "3"?
Solution: There are 4 outcomes that satisfy our condition (at least 3): {3, 4, 5, 6}. The probability of each outcome is 1/6. The probability of getting at least a "3" is:

$$P=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{2}{3}$$

Combination of independent and mutually exclusive events

Many probability problems contain combination of both independent and mutually exclusive events. To solve those problems it is important to identify all events and their types. One of the typical problems can be presented in a following general form:

Q: If the probability of a certain event is p, what is the probability of it occurring k times in n-time sequence?
(Or in English, what is the probability of getting 3 heads while tossing a coin 8 times?)
Solution: All events are independent. So, we can say that:

$$P'=p^k*(1-p)^{n-k} \quad (1)$$

But it isn't the right answer. It would be right if we specified exactly each position for events in the sequence. So, we need to take into account that there are more than one outcomes. Let's consider our example with a coin where "H" stands for Heads and "T" stands for Tails:
HHHTTTTT and HHTTTTTH are different mutually exclusive outcomes but they both have 3 heads and 5 tails. Therefore, we need to include all combinations of heads and tails. In our general question, probability of occurring event k times in n-time sequence could be expressed as:

$$P=C_k^n*p^k*(1-p)^{n-k} \quad (2)$$

In the example with a coin, right answer is $P=C_3^8*0.5^3*0.5^5=C_3^8*0.5^8$

Example #1
Q.: If the probability of raining on any given day in Atlanta is 40 percent, what is the probability of raining on exactly 2 days in a 7-day period?
Solution: We are not interested in the exact sequence of event and thus apply formula #2:

$$P=C_2^7*0.4^2*0.6^5$$

A few ways to approach a probability problem

There are a few typical ways that you can use for solving probability questions. Let's consider example, how it is possible to apply different approaches:

Example #1
Q: There are 8 employees including Bob and Rachel. If 2 employees are to be randomly chosen to form a committee, what is the probability that the committee includes both Bob and Rachel?
Solution:

1) combinatorial approach: The total number of possible committees is $N = C^8_2$. The number of possible committee that includes both Bob and Rachel is $n = 1$.

$$P = \frac{n}{N} = \frac{1}{C^8_2} = \frac{1}{28}$$

2) reversal combinatorial approach: Instead of counting probability of occurrence of certain event, sometimes it is better to calculate the probability of the opposite and then use formula p = 1 - q. The total number of possible committees is $N = C^8_2$. The number of possible committee that does not includes both Bob and Rachel is:

$m = C^6_2 + 2*C^6_1$ where,

C^6_2 - the number of committees formed from 6 other people.

$2*C^6_1$ - the number of committees formed from Rob or Rachel and one out of 6 other people.

$$P = 1 - \frac{m}{N} = 1 - \frac{C^6_2 + 2*C^6_1}{C^8_2}$$

$$P = 1 - \frac{15 + 2*6}{28} = 1 - \frac{27}{28} = \frac{1}{28}$$

3) probability approach: The probability of choosing Bob or Rachel as a first person in committee is 2/8. The probability of choosing Rachel or Bob as a second person when first person is already chosen is 1/7. The probability that the committee includes both Bob and Rachel is.

$$P = \frac{2}{8} * \frac{1}{7} = \frac{2}{56} = \frac{1}{28}$$

4) reversal probability approach: We can choose any first person. Then, if we have Rachel or Bob as first choice, we can choose any other person out of 6 people. If we have neither Rachel nor Bob as first choice, we can choose any person out of remaining 7 people. The probability that the committee includes both Bob and Rachel is.

$$P = 1 - \left(\frac{2}{8} * \frac{6}{7} + \frac{6}{8} * 1\right) = \frac{2}{56} = \frac{1}{28}$$

Example #2
Q: Given that there are 5 married couples. If we select only 3 people out of the 10, what is the probability that none of them are married to each other?
Solution:

1) combinatorial approach:

C^5_3 - we choose 3 couples out of 5 couples.

C^2_1 - we chose one person out of a couple.

$(C^2_1)^3$ - we have 3 couple and we choose one person out of each couple.

C^{10}_3 - the total number of combinations to choose 3 people out of 10 people.

$$p = \frac{C_3^5 * (C_1^2)^3}{C_3^{10}} = \frac{10*8}{10*3*4} = \frac{2}{3}$$

2) Reversal combinatorial approach: In this example reversal approach is a bit shorter and faster.

C_1^5 - we choose 1 couple out of 5 couples.

C_1^8 - we chose one person out of remaining 8 people.

C_3^{10} - the total number of combinations to choose 3 people out of 10 people.

$$p = 1 - \frac{C_1^5 * C_1^8}{C_3^{10}} = 1 - \frac{5*8}{10*3*4} = \frac{2}{3}$$

3) probability approach:

1st person: $\frac{10}{10} = 1$ - we choose any person out of 10.

2nd person: $\frac{8}{9}$ - we choose any person out of 8=10-2(one couple from previous choice)

3rd person: $\frac{6}{8}$ - we choose any person out of 6=10-4(two couples from previous choices).

$$p = 1 * \frac{8}{9} * \frac{6}{8} = \frac{2}{3}$$

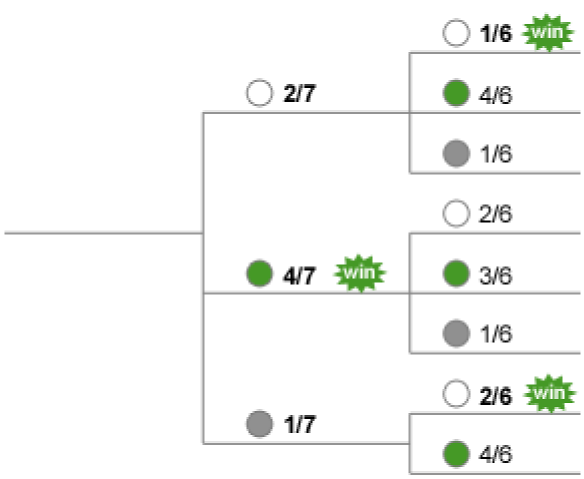
Probability tree

Sometimes, at 700+ level you may see complex probability problems that include conditions or restrictions. For such problems it could be helpful to draw a probability tree that include all possible outcomes and their probabilities.

Example #1

Q: Julia and Brian play a game in which Julia takes a ball and if it is green, she wins. If the first ball is not green, she takes the second ball (without replacing first) and she wins if the two balls are white or if the first ball is gray and the second ball is white. What is the probability of Julia winning if the jar contains 1 gray, 2 white and 4 green balls?

Solution: Let's draw all possible outcomes and calculate all probabilities.



Now, It is pretty obvious that the probability of Julia's win is:

$$P = \frac{4}{7} + \frac{2}{7} * \frac{1}{6} + \frac{1}{7} * \frac{2}{6} = \frac{2}{3}$$

Tips and Tricks: Symmetry

Symmetry sometimes lets you solve seemingly complex probability problem in a few seconds. Let's consider an example:

Example #1
Q: There are 5 chairs. Bob and Rachel want to sit such that Bob is always left to Rachel. How many ways it can be done?
Solution: Because of symmetry, the number of ways that Bob is left to Rachel is exactly 1/2 of all possible ways:

$$N = \frac{1}{2} * P_2^5 = 10$$

Practice from the GMAT Official Guide:

The Official Guide, 12th Edition: DT #4; DT #7; PS #12; PS #67; PS #105; PS #158; PS #174; PS #214; DS #3; DS #107;
The Official Guide, Quantitative 2th Edition: PS #79; PS #160;
The Official Guide, 11th Edition: DT #4; DT #7; PS #10; PS #64; PS #173; PS #217; PS #231; DS #82; DS #114;

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Resources

Probability DS problems: [search]
Probability PS problems: [search]

Walker's post with Combinatorics/probability problems: [Combinatorics/probability Problems]
Bullet's post with probability problems: [Combined Probability Questions]

Combinations & Permutations

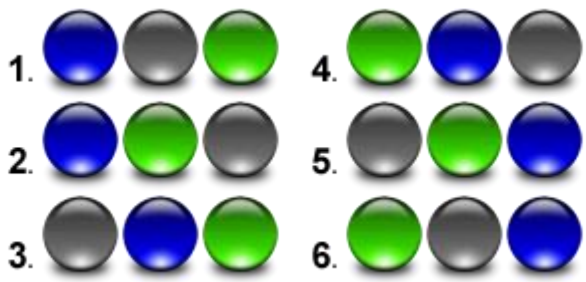
Definition

Combinatorics is the branch of mathematics studying the enumeration, combination, and permutation of sets of elements and the mathematical relations that characterize their properties.

Enumeration

Enumeration is a method of counting all possible ways to arrange elements. Although it is the simplest method, it is often the fastest method to solve hard GMAT problems and is a pivotal principle for any other combinatorial method. In fact, combination and permutation is shortcuts for enumeration. The main idea of enumeration is writing down all possible ways and then count them. Let's consider a few examples:

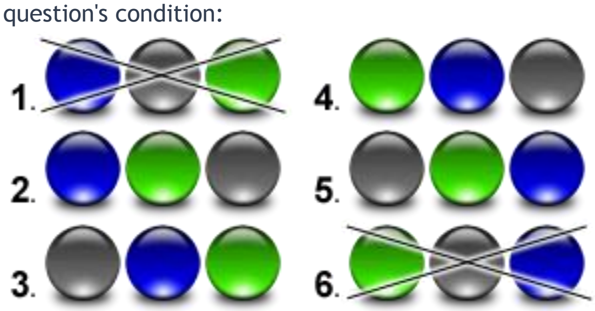
Example #1
Q:. There are three marbles: 1 blue, 1 gray and 1 green. In how many ways is it possible to arrange marbles in a row?
Solution: Let's write out all possible ways:



Answer is 6.

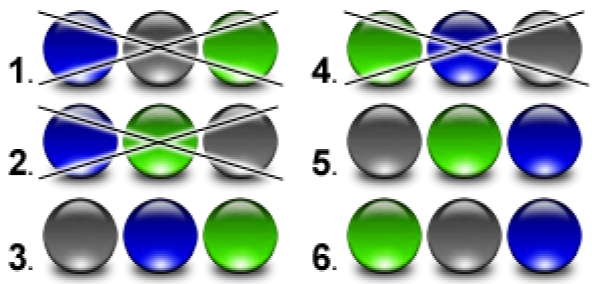
In general, the number of ways to arrange n different objects in a row

Example #2
Q:. There are three marbles: 1 blue, 1 gray and 1 green. In how many ways is it possible to arrange marbles in a row if blue and green marbles have to be next to each other?
Solution: Let's write out all possible ways to arrange marbles in a row and then find only arrangements that satisfy question's condition:



Answer is 4.

Example #3
Q:. There are three marbles: 1 blue, 1 gray and 1 green. In how many ways is it possible to arrange marbles in a row if gray marble have to be left to blue marble?
Solution: Let's write out all possible ways to arrange marbles in a row and then find only arrangements that satisfy question's condition:



Answer is 3.

Arrangements of n different objects

Enumeration is a great way to count a small number of arrangements. But when the total number of arrangements is large, enumeration can't be very useful, especially taking into account GMAT time restriction. Fortunately, there are some methods that can speed up counting of all arrangements.

The number of arrangements of n different objects in a row is a typical problem that can be solve this way:

1. How many objects we can put at 1st place? n.
2. How many objects we can put at 2nd place? n - 1. We can't put the object that already placed at 1st place.
.....
n. How many objects we can put at n-th place? 1. Only one object remains.

Therefore, the total number of arrangements of n different objects in a row is

$$N = n*(n-1)*(n-2)....2*1 = n!$$

Combination

A combination is an unordered collection of k objects taken from a set of n distinct objects. The number of ways how we can choose k objects out of n distinct objects is denoted as:

$$C^n_k$$

knowing how to find the number of arrangements of n distinct objects we can easily find formula for combination:

1. The total number of arrangements of n distinct objects is n!
2. Now we have to exclude all arrangements of k objects (k!) and remaining (n-k) objects ((n-k)!) as the order of chosen k objects and remained (n-k) objects doesn't matter.

$$C^n_k = \frac{n!}{k!(n-k)!}$$

Permutation

A permutation is an ordered collection of k objects taken from a set of n distinct objects. The number of ways how we can choose k objects out of n distinct objects is denoted as:

$$P^n_k$$

knowing how to find the number of arrangements of n distinct objects we can easily find formula for combination:

1. The total number of arrangements of n distinct objects is n!
2. Now we have to exclude all arrangements of remaining (n-k) objects ((n-k)!) as the order of remained (n-k) objects doesn't matter.

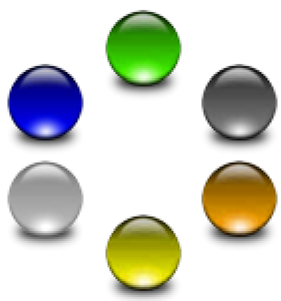
$$P_k^n = \frac{n!}{(n-k)!}$$

If we exclude order of chosen objects from permutation formula, we will get combination formula:

$$\frac{P_k^n}{k!} = C_k^n$$

Circular arrangements

Let's say we have 6 distinct objects, how many relatively different arrangements do we have if those objects should be placed in a circle.



The difference between placement in a row and that in a circle is following: if we shift all object by one position, we will get different arrangement in a row but the same relative arrangement in a circle. So, for the number of circular arrangements of n objects we have:

$$R = \frac{n!}{n} = (n-1)!$$

Tips and Tricks

Any problem in Combinatorics is a counting problem. Therefore, a key to solution is a way how to count the number of arrangements. It sounds obvious but a lot of people begin approaching to a problem with thoughts like "Should I apply C- or P- formula here?". Don't fall in this trap: define how you are going to count arrangements first, realize that your way is right and you don't miss something important, and only then use C- or P- formula if you need them.

Resources

Combinatorics DS problems: [search]
Combinatorics PS problems: [search]

Walker's post with Combinatorics/probability problems: [Combinatorics/probability Problems]

Sequences & Progressions

Definition

Sequence: It is an ordered list of objects. It can be finite or infinite. The elements may repeat themselves more than once in the sequence, and their ordering is important unlike a set

Arithmetic Progressions

Definition
It is a special type of sequence in which the difference between successive terms is constant.

General Term
$a_n = a_{n-1} + d = a_1 + (n-1)d$
a_i is the ith term
d is the common difference
a_1 is the first term

Defining Properties
Each of the following is necessary & sufficient for a sequence to be an AP :

- $a_i - a_{i-1} =$ Constant
- If you pick any 3 consecutive terms, the middle one is the mean of the other two
- For all i,j > k >= 1 : $\frac{a_i - a_k}{i-k} = \frac{a_j - a_k}{j-k}$

Summation
The sum of an infinite AP can never be finite except if $a_1 = 0$ & $d = 0$
The general sum of a n term AP with common difference d is given by $\frac{n}{2}(2a+(n-1)d)$
The sum formula may be re-written as $n*Avg(a_1, a_n) = \frac{n}{2}*(FirstTerm + LastTerm)$

Examples

1. All odd positive integers : {1,3,5,7,...} $a_1 = 1, d = 2$
2. All positive multiples of 23 : {23,46,69,92,...} $a_1 = 23, d = 23$
3. All negative reals with decimal part 0.1 : {-0.1,-1.1,-2.1,-3.1,...} $a_1 = -0.1, d = -1$

Geometric Progressions

Definition
It is a special type of sequence in which the ratio of consecutive terms is constant

General Term
$b_n = b_{n-1} * r = a_1 * r^{n-1}$
b_i is the ith term
r is the common ratio

b_1 is the first term

Defining Properties
Each of the following is necessary & sufficient for a sequence to be an AP :

- $\frac{b_i}{b_{i-1}} =$ Constant
- If you pick any 3 consecutive terms, the middle one is the geometric mean of the other two
- For all i,j > k >= 1 : $(\frac{b_i}{b_k})^{j-k} = (\frac{b_j}{b_k})^{i-k}$

Summation
The sum of an infinite GP will be finite if absolute value of r < 1

The general sum of a n term GP with common ratio r is given by $b_1 * \frac{r^n - 1}{r-1}$

If an infinite GP is summable (|r|<1) then the sum is $\frac{b_1}{1-r}$

Examples

1. All positive powers of 2 : {1,2,4,8,...} $b_1 = 1, r = 2$
2. All positive odd and negative even numbers : {1,-2,3,-4,...} $b_1 = 1, r = -1$
3. All negative powers of 4 :
{1/4,1/16,1/64,1/256,...} $b_1 = 1/4, r = 1/4, sum = \frac{1/4}{(1-1/4)} = (1/3)$

Harmonic Progressions

Definition
It is a special type of sequence in which if you take the inverse of every term, this new sequence forms an AP

Important Properties
Of any three consecutive terms of a HP, the middle one is always the harmonic mean of the other two, where the harmonic mean (HM) is defined as :
$\frac{1}{2} * (\frac{1}{a} + \frac{1}{b}) = \frac{1}{HM(a,b)}$
Or in other words :
$HM(a,b) = \frac{2ab}{a+b}$

APs, GPs, HPs : Linkage

Each progression provides us a definition of "mean" :

Arithmetic Mean : $\frac{a+b}{2}$ OR $\frac{a1 + .. + an}{n}$

Geometric Mean : $\sqrt{ab}$ OR $(a1 * .. * an)^{\frac{1}{n}}$

Harmonic Mean : $\frac{2ab}{a+b}$ OR $\frac{n}{\frac{1}{a1} + .. + \frac{1}{an}}$

For all non-negative real numbers : AM >= GM >= HM

In particular for 2 numbers : AM * HM = GM * GM

Example :
Let a=50 and b=2,
then the AM = (50+2)*0.5 = 26 ;
the GM = sqrt(50*2) = 10 ;
the HM = (2*50*2)/(52) = 3.85
AM > GM > HM
AM*HM = 100 = GM^2

Misc. Notes

A subsequence (any set of consecutive terms) of an AP is an AP

A subsequence (any set of consecutive terms) of a GP is a GP

A subsequence (any set of consecutive terms) of a HP is a HP

If given an AP, and I pick out a subsequence from that AP, consisting of the terms $a_{i1}, a_{i2}, a_{i3}, \dots$ *such that* $i1, i2, i3$ *are in AP then the new subsequence will also be an AP*

For Example : Consider the AP with $a_1 = 1, d = 2$ {1,3,5,7,9,11,...}, so a_n=1+2*(n-1)=2n-1
Pick out the subsequence of terms $a_5, a_{10}, a_{15}, \dots$
New sequence is {9,19,29,...} which is an AP with $a_1 = 9$ and $d = 10$

If given a GP, and I pick out a subsequence from that GP, consisting of the terms $b_{i1}, b_{i2}, b_{i3}, \dots$ *such that* $i1, i2, i3$ *are in AP then the new subsequence will also be a GP*

For Example : Consider the GP with $b_1 = 1, r = 2$ {1,2,4,8,16,32,...}, so b_n=2^(n-1)
Pick out the subsequence of terms $b_2, b_4, b_6, \dots$
New sequence is {4,16,64,...} which is a GP with $b_1 = 4$ and $r = 4$

The special sequence in which each term is the sum of previous two terms is known as the Fibonacci sequence. It is neither an AP nor a GP. The first two terms are 1. {1,1,2,3,5,8,13,...}

In a finite AP, the mean of all the terms is equal to the mean of the middle two terms if n is even and the middle term if n is even. In either case this is also equal to the mean of the first and last terms

Some examples

Example 1
A coin is tossed repeatedly till the result is a tails, what is the probability that the total number of tosses is less than or equal to 5 ?

Solution
P(<=5 tosses) = P(1 toss)+...+P(5 tosses) = P(T)+P(HT)+P(HHT)+P(HHHT)+P(HHHHT)
We know that P(H)=P(T)=0.5
So Probability = 0.5 + 0.5^2 + ... + 0.5^5
This is just a finite GP, with first term = 0.5, n=5 and ratio = 0.5. Hence :

Probability = $0.5 * \frac{1-0.5^5}{1-0.5} = \frac{1}{2} * \frac{\frac{31}{32}}{\frac{1}{2}} = \frac{31}{32}$

Example 2
In an arithmetic progression a1,a2,...,a22,a23, the common difference is non-zero, how many terms are greater than 24 ?
(1) a1 = 8
(2) a12 = 24

Solution
(1) a1=8, does not tell us anything about the common difference, so impossible to say how many terms are greater than 24
(2) a12=24, and we know common difference is non-zero. So either all the terms below a12 are greater than 24 and the terms above it less than 24 or the other way around. In either case, there are exactly 11 terms either side of a12. Sufficient
Answer is B

Example 3
For positive integers a,b (a<b) arrange in ascending order the quantities a, b, sqrt(ab), avg(a,b), 2ab/(a+b)

Solution
Using the inequality AM>=GM>=HM, the solution is :
a <= 2ab/(a+b) <= Sqrt(ab) <= Avg(a,b) <= b

Example 4
For every integer k from 1 to 10, inclusive, the kth term of a certain sequence is given by (-1)^(k+1) *(1/2^k). If T is the sum of the first 10 terms in the sequence then T is

a)greater than 2
b)between 1 and 2
c)between 1/2 and 1
d)between 1/4 and 1/2
e)less than 1/4.

Solution
The sequence given has first term 1/2 and each subsequent term can be obtained by multiplying with -1/2. So it is a GP. We can use the GP summation formula

$$S = b\frac{1-r^n}{1-r} = \frac{1}{2}*\frac{1-(-1/2)^{10}}{1-(-1/2)} = \frac{1}{3}*\frac{1023}{1024}$$

1023/1024 is very close to 1, so this sum is very close to 1/3
Answer is d

Example 5
The sum of the fourth and twelfth term of an arithmetic progression is 20. What is the sum of the first 15 terms of the arithmetic progression?
A. 300
B. 120
C. 150
D. 170
E. 270

Solution

$$a_4 + a_12 = 20$$
$$a_4 = a_1 + 3d, a_12 = a_1 + 11d$$
$$2a_1 + 14d = 20$$

Now we need the sum of first 15 terms, which is given by:

$$\frac{15}{2}(2a_1 + (15-1)d) = \frac{15}{2}*(2a_1 + 14d) = 150$$

Answer is (c)

Additional Exercises

- toughest-progression-questions-99380.html
- arithmetic-progression-82035.html
- PS Sequence Questions
- DS Sequence questions

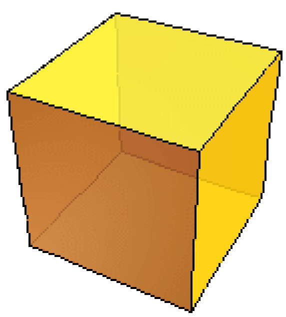
3-D Geometries

Scope

The GMAT often tests on the knowledge of the geometries of 3-D objects such cylinders, cones, cubes & spheres. The purpose of this document is to summarize some of the important ideas and formulae and act as a useful cheat sheet for such questions

Cube

Image:



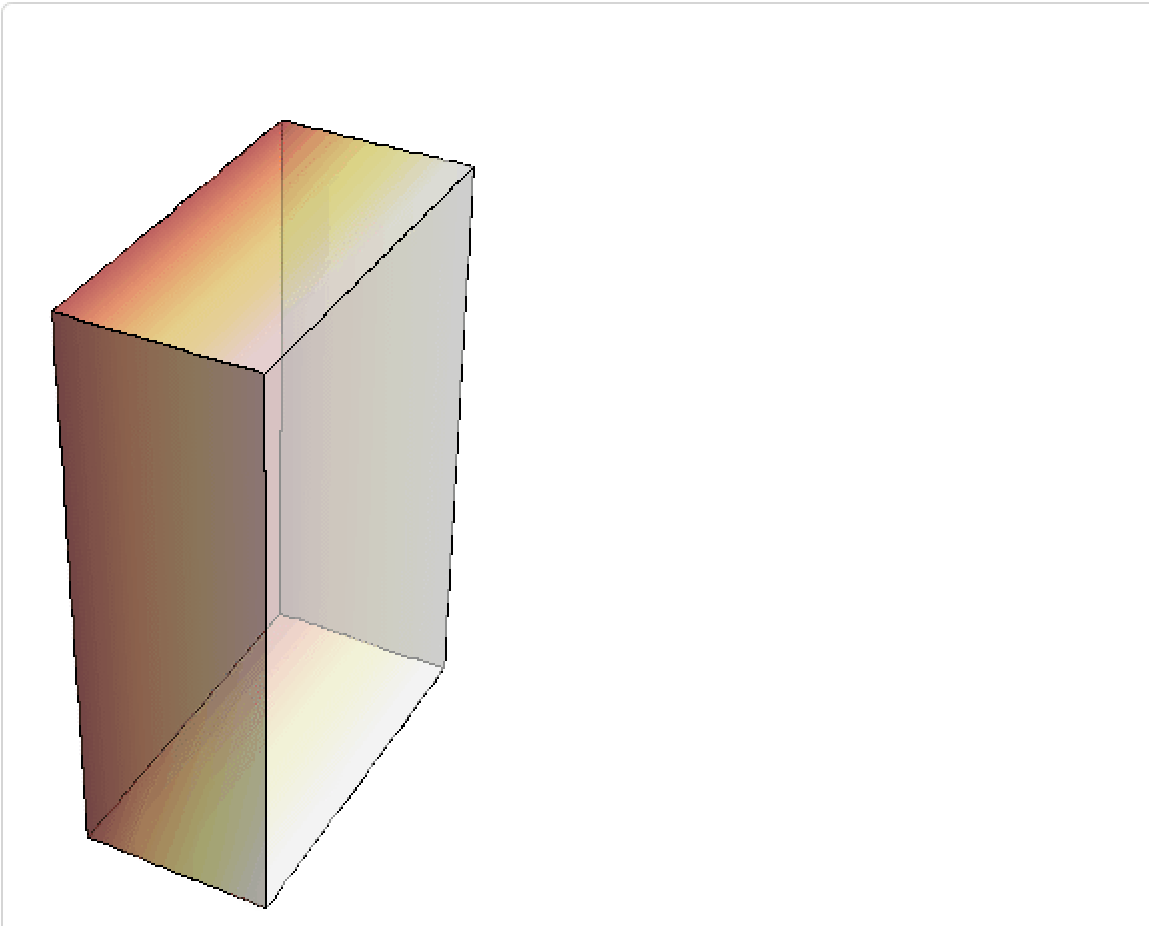
Cube

WolframAlpha

A cube is the 3-D generalization of a square, and is characterized by the length of the side, a. Important results include:

- Volume = a^3
- Surface Area = $6a^2$
- Diagonal Length = $\sqrt{3}a$

Cuboid

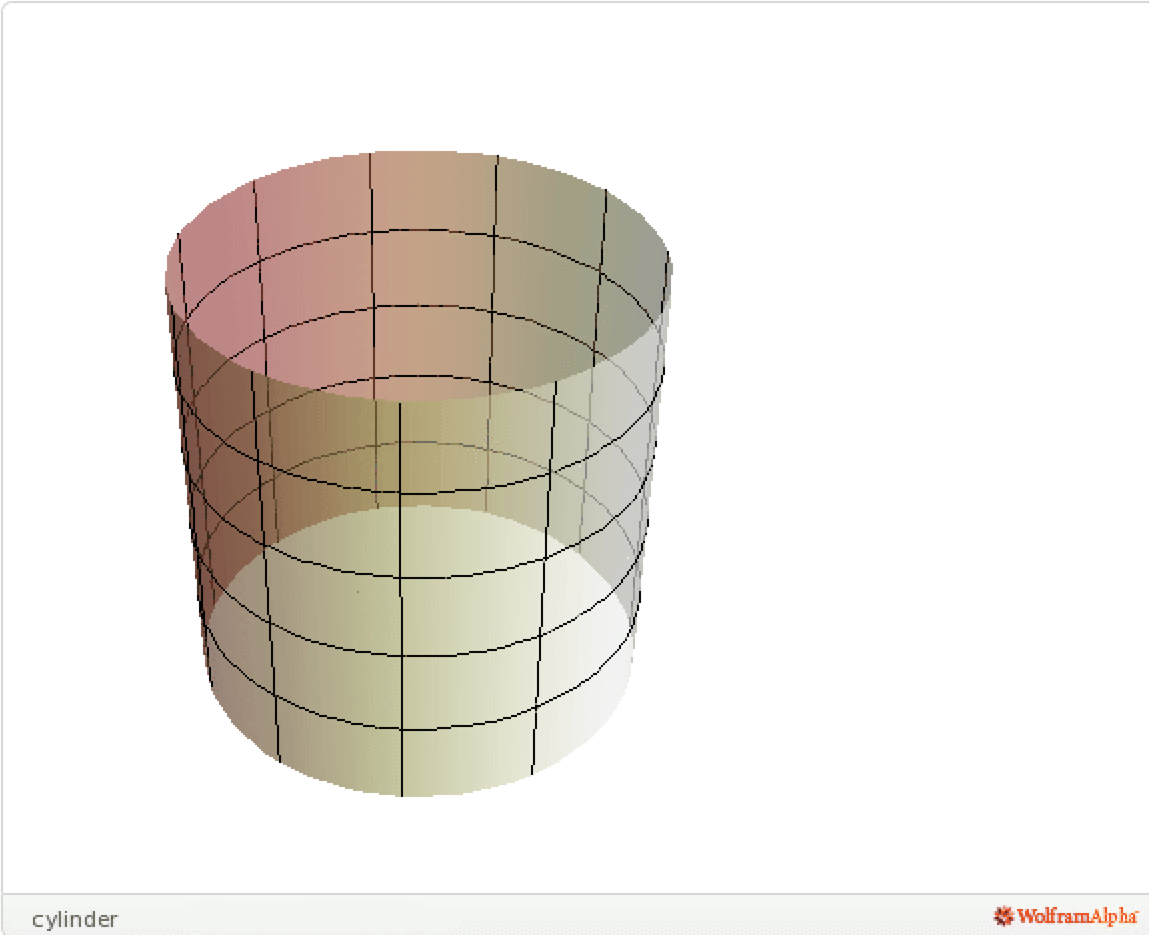


Cuboid

A cube is the 3-D generalization of a rectangle, and is characterized by the length of its sides, a, b, c. Important results include:

- Volume = abc
- Surface Area = $2(ab+bc+ca)$
- Diagonal Length = $\sqrt{a^2+b^2+c^2}$

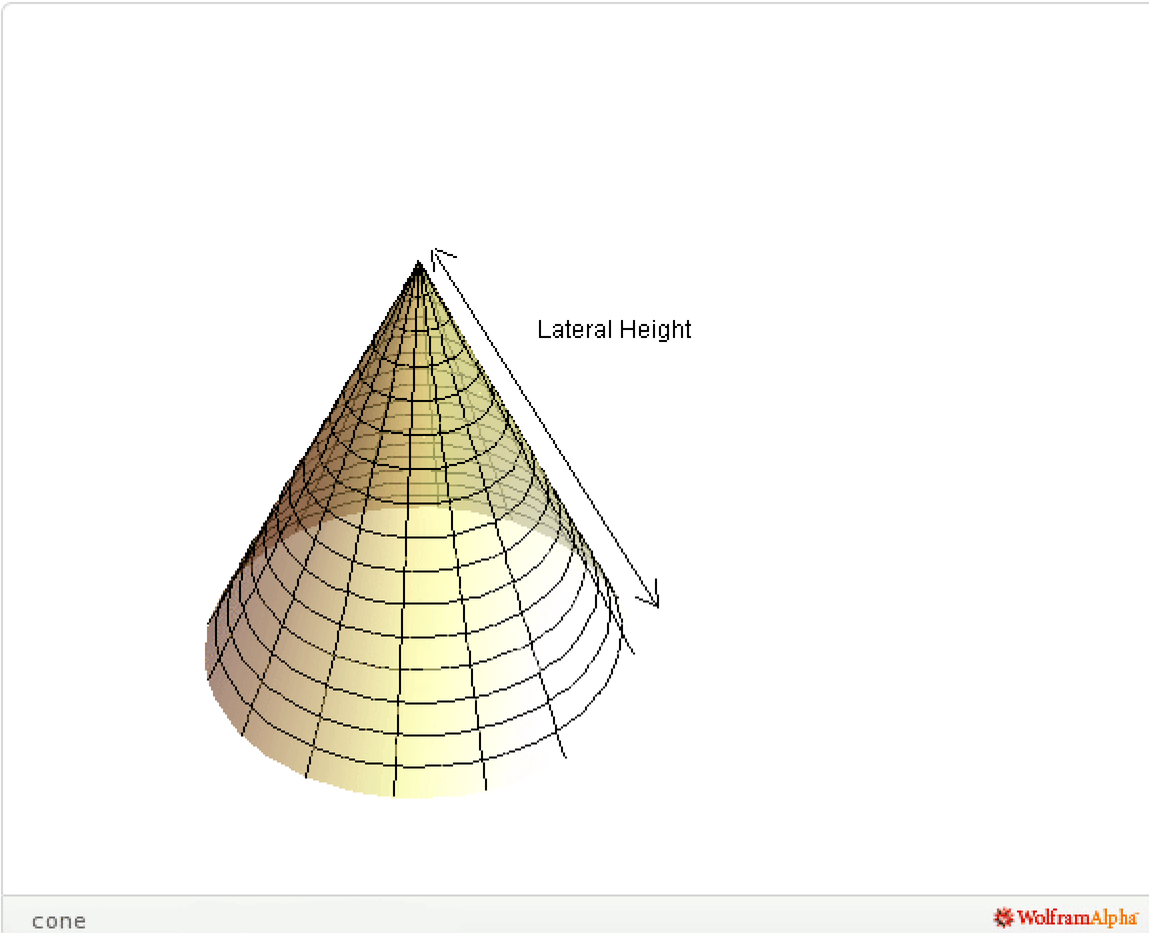
Cylinder



A cylinder is a 3-D object formed by rotating a rectangular sheet along one of its sides. It is characterized by the radius of the base, r, and the height, h . Important results include:

- Volume = $\pi r^2 h$
- Outer surface area w/o bases = $2\pi r h$
- Outer surface area including bases = $2\pi r(r+h)$

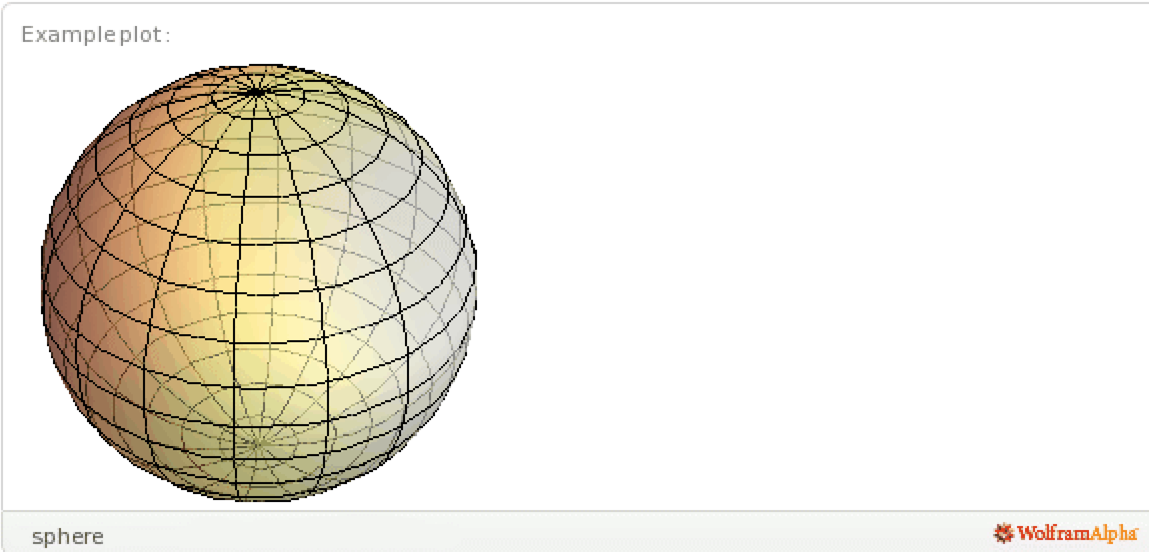
Cone



A cone is a 3-D object obtained by rotating a right angled triangle around one of its sides. It is characterized by the radius of its base, r, and the height, h. The hypotenuse of the triangle formed by the height and the radius (running along the diagonal side of the cone), is known as it lateral height, $l=\sqrt{r^2+h^2}$. Important results include:

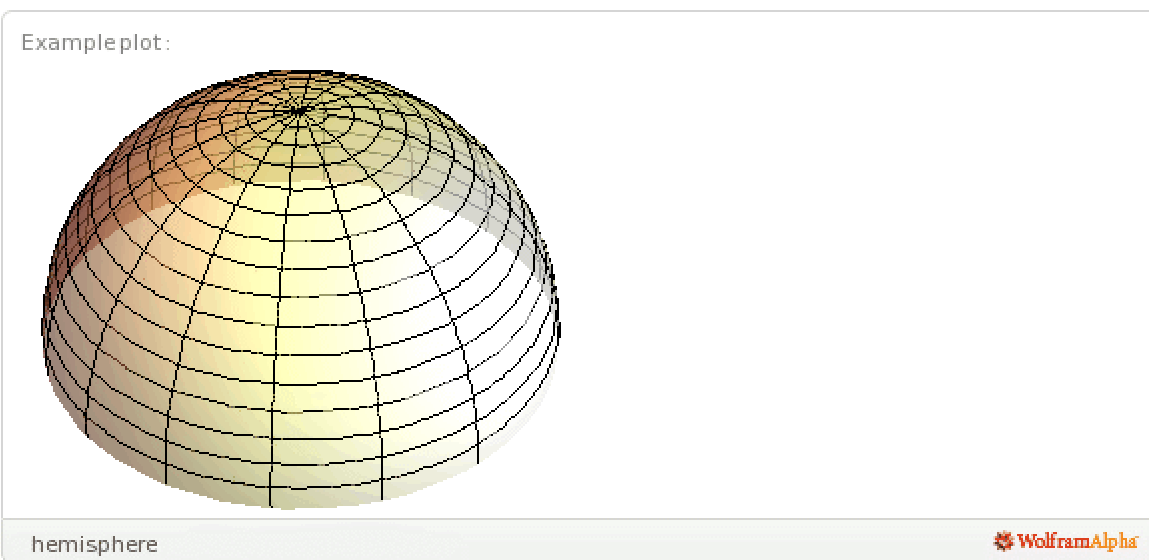
- Volume = $\frac{1}{3}\pi r^2 h$
- Outer surface area w/o base = $\pi r l=\pi r\sqrt{r^2+h^2}$
- Outer surface area including base = $\pi r(r+l)=\pi r(r+\sqrt{r^2+h^2})$

Sphere



A sphere is a 3-D generalization of a circle. It is characterized by its radius, r. Important results include:

- Volume = $\frac{4}{3}\pi r^3$
- Surface Area= $4\pi r^2$



A hemisphere is a sphere cut in half and is also characterized by its radius r. Important results include:

Volume = $\frac{2}{3}\pi r^3$

Surface Area w/o base = $2\pi r^2$

Surface Area with base = $3\pi r^2$

Some simple configurations

These may appear in various forms on the GMAT, and are good practice to derive on one's own :

1. Sphere inscribed in cube of side a : Radius of sphere is $\frac{a}{2}$
2. Cube inscribed in sphere of radius r : Side of cube is $\frac{2r}{\sqrt{3}}$
3. Cylinder inscribed in cube of side a : Radius of cylinder is $\frac{a}{2}$; Height a
4. Cone inscribed in cube of side a : Radius of cone is $\frac{a}{2}$; Height a
5. Cylinder of radius r in sphere of radius R ($R>r$) : Height of cylinder is $2\sqrt{R^2-r^2}$

Examples

Example 1 : A certain right circular cylinder has a radius of 5 inches. There is oil filled in this cylinder to the height of 9 inches. If the oil is poured completely into a second right cylinder, then it will fill the second cylinder to a height of 4 inches. What is the radius of the second cylinder, in inches?

A. 6
B. 6.5
C. 7
D. 7.5
E. 8

Solution : The volume of the liquid is constant.
Initial volume = $\pi * 5^2 * 9$
New volume = $\pi * r^2 * 4$

$$\pi * 5^2 * 9 = \pi * r^2 * 4$$
$$r = (5*3)/2 = 7.5$$

Answer is (d)

Example 2 : A spherical balloon has a volume of 972 π cubic cm, what is the surface area of the balloon in sq.

cm?

A) 324

B) 729

C) 243 π

D) 324 π

E) 729 π

Solution: $V = \frac{4}{3}\pi r^3$

$A = 4\pi r^2 = 4 * \pi * 9^2 = 324\pi$

Answer (d)

Example 3 : A cube of side 5cm is painted on all its side. If it is sliced into 1 cubic centime cubes, how many 1 cubic centimeter cubes will have exactly one of their sides painted?

A. 9

B. 61

C. 98

D. 54

E. 64

Solution: Notice that the new cubes will be each of side 1Cm. So on any face of the old cube there will be 5x5=25 of the smaller cubes. Of these, any smaller cube on the edge of the face will have 2 faces painted (one for every face shared with the bigger cube). The number of cubes that have exactly one face painted are all except the ones on the edges. Number on the edges are 16, so 9 per face.

There are 6 faces, hence 6*9=54 smaller cubes with just one face painted.

Answer is (d)

Example 4 : What is the surface area of the cuboid C ?

(1) The length of the diagonal of C is 5

(2) The sum of the sides of C is 10

Solution: Let the sides of cuboid C be x, y, z

We know that the surface area is given be $2(xy + yz + zx)$

(1) : Diagonal = $\sqrt{x^2 + y^2 + z^2} = 5$. Not sufficient to know the area

(2) : Sum of sides = $x+y+z=10$. Not sufficient to know the area

(1+2) : Note the identity $(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)$

Now we clearly have enough information.

$2(xy+yz+zx)=10^2-5^2=75$

Sufficient

Answer is (c)

Sample Problems

Sphere & Cube

Sphere & Cylinder

Cylinder & Cuboid

Cylinder & Cuboid II

Cylinder

Cube

Cube II

Cone

Cube III

Cylinder

Hemisphere