11. You are given a hundred-sided die. Every time you roll you can choose to either get paid the dollar amount of that roll or pay one dollar for one more roll. What is the expected value of the game?

Denote the roll of the die as X. The expected value of a single throw of the die is

$$\mathbb{E}(X) = \frac{1}{100}\sum_{n=1}^{100} n = \frac{101}{2} = 50.5$$

so the expected value of a throw with a reroll is 49.5. The strategy is therefore such that if I get 49 or lower, I will choose to pay the dollar and roll again. Otherwise I will stick to what I've got. Then the expected value of the game is

$$\begin{aligned}\mathbb{E}(\text{game}) &= P\left(X \leq 49\right)\left(\mathbb{E}(X) - 1\right) + P\left(X \geq 50\right)\mathbb{E}\left(X|X \geq 50\right) \\ &= \frac{49 \cdot 49.5}{100} + \frac{51}{100}\frac{1}{100}\sum_{n=50}^{100} n = 62.5.\end{aligned}$$

12. You and I play a game with two dice, one ten-sided and one six-sided. You guess the sum of the numbers after we roll them. If your guess is correct, you get the sum of the numbers in dollars, else you get nothing. How would you make the best guess?

Denote a roll of the ten sided and six sided dice respectively as X and Y. The probability $P(X+Y=n)$ is just the number of possible combinations of $X+Y=n$ divided by the total number of combinations, which is 60. From 1 to 7 the number of combinations is the same as with two six-sided dice, but to get 8, the possible pairs are $(X,Y) = (7,1), (6,2), \ldots, (2,6)$, i.e. total 6 combinations. Similarly, to get 9 we have pairs $(8,1), (7,2), \ldots, (3,6)$, which gives 6 combinations etc. up to total sum of 11. To get 12, we have pairs $(10,2), (9,3), \ldots, (6,6)$, which is total 5 pairings, and the number just decreases for totals higher than this. So the total number of combinations is $(0,1,2,3,4,5,6,6,6,6,6,5,4,3,2,1)$ for a total sum of $(1,2,\ldots)$. The highest probabilities occur for 7,8,9,10 and 11, so we should choose 11 to maximize our winnings.

13. (CHECK) Is it possible to change the numbers on two six-sided dice to other positive numbers so that the probability distribution of their sum remains unchanged?

Solution

14. If you toss three coins, what is the probability of getting at least two heads?

Number of all possible head, tail combinations is $2^3 = 8$. The number of ways of choosing two heads and one tail is $\binom{3}{2} = 3$ and then we have the case of three heads, which gives the final probability

$$P = \frac{3+1}{8} = \frac{1}{2}.$$

15. If you toss a coin six times, what is the expected number of heads?

The total number of possibilities is 2^6. The number of ways of choosing n heads is $\binom{6}{n}$, which gives the probability distribution for the number of heads,

$$p_n \doteq \binom{6}{n} 2^{-6}$$

and the expected number of heads can be computed by reorganizing the sum over the binomial coefficients as follows:

$$\begin{aligned} E &= \sum_{n=0}^{6} n p_n = \sum_{n=0}^{6} n \binom{6}{n} 2^{-6} = 2^{-6} 6! \sum_{n=0}^{6} \frac{n}{(6-n)!n!} \\ &= 2^{-6} 6! \sum_{n=1}^{6} \frac{1}{(6-n)!(n-1)!} = 2^{-6} 6! \sum_{k=0}^{5} \frac{1}{(5-k)!k!} = \frac{6}{2} \sum_{k=0}^{5} \frac{5!}{(5-k)!k!} 2^{-5} \\ &= 3 \sum_{n=0}^{5} n \binom{5}{n} 2^{-5} = 3 \end{aligned}$$

16. (CHECK) If you drop four coins on the table, given that one is heads, what is the probability that the rest are tails?

Solution

17. You flip four coins and I give you a dollar for each heads.
A) How much would you pay to play this game?
B) What if I give you the option to re-flip one of the coins?
C) What if I have the option to re-flip one of the coins?

A)
Denote by X the number of heads. Denote $p_n = P(X = n) = \binom{4}{n} 2^{-4}$. Expected payoff for this game is then